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Jimm, a Fundamental Involution

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Dedicated to Yılmaz Akyıldız, who shared our enthusiasm about jimm

Abstract

We study the involution of the real line induced by the outer automorphism of the extended modular group PGL(2,Z). This 'modular' involution is discontinuous at rationals but satisfies a surprising collection of functional equations. It preserves the set of real quadratic irrationalities mapping them in a highly non-obvious way to each other. It commutes with the Galois action on real quadratic irrationals.

More generally, it preserves set-wise the orbits of the modular group, thereby inducing an involution of the moduli space of real rank-two lattices. It induces a duality of Beatty partitions of the set of positive integers.

This involution conjugates (though not topologically) the Gauss' continued fraction map to an intermittent dynamical system on the unit interval with an infinite invariant measure. The transfer operator (resp. the functional equation) naturally associated to this dynamical system is closely related to the Mayer transfer operator (resp. the Lewis' functional equation).

We give a description of this involution as the boundary action of a certain automorphism of the infinite trivalent tree. We prove that its derivative exists and vanishes almost everywhere. It is conjectured that algebraic numbers of degree at least three are mapped to transcendental numbers under this involution.

1 Introduction

It is (it seems not very well-) known that the group $\mathrm{PGL}_2(\mathbf{Z})$ has an involutive outer automorphism, which was discovered by Dyer in the late 70's [14]. It would

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be very strange if this automorphism had no manifestations in myriad contexts where $\mathrm{PGL}_2(\mathbf{Z})$ or its subgroups play a major role. Our aim in this paper is to elucidate one of these manifestations, which appears to have remained in obscurity until now. This may be because the conventional arithmetic, algebraic and geometric structures are not 'respected' by this involution (i.e. it sends parabolics to hyperbolics) which makes its study all the more appealing. Being thus exotic in several ways, we denote it - and some other involutions it gives rise to - by the letter[1] ج (read as: "jimm"), hoping that this notation will help to keep track of its manifestations.

Let $\widehat{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$. The manifestation in question of ج is a map $\text{ج}_{\mathbf{R}} : \widehat{\mathbf{R}} \to \widehat{\mathbf{R}}$. Denoting the continued fractions in the usual way

$$[n_0, n_1, n_2, \dots] = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\cdots}}},$$

one has, for an irrational number $[n_0, n_1, n_2, \dots]$ with $n_0, n_1, \dots \geq 2$,

$$\text{ج}_{\mathbf{R}}([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots] \tag{1}$$

where 1_k is the sequence $1, 1, \dots, 1$ of length k. This formula remains valid for $n_0, n_1, \dots \geq 1$, if the emerging 1_{-1}'s are eliminated in accordance with the rule $[\dots m, 1_{-1}, n, \dots] = [\dots m + n - 1, \dots]$ and 1_0 with the rule $[\dots m, 1_0, n, \dots] = [\dots m, n, \dots]$. See page 5 below for some examples.

It is possible to extend this definition of $\text{ج}_{\mathbf{R}}$ to all of $\widehat{\mathbf{R}}$. If we ignore rationals and the noble numbers (i.e. numbers in the $\mathrm{PGL}_2(\mathbf{Z})$-orbit of the golden section, $\Phi := (1 + \sqrt{5})/2$), then $\text{ج}_{\mathbf{R}}$ becomes an involution. It is well-defined and continuous at irrationals, but two-valued and discontinuous at rationals (Theorem 16). Although it is impossible to draw its graph, we will give (see page 36) a boxed-graph to indicate where the graph lies. One can choose one among the two values at rational arguments so that the function becomes upper semicontinuous. The amount of jump of $\text{ج}_{\mathbf{R}}$ at $q \in \mathbf{Q}$ provides a canonical (signed) measure of complexity of a rational number.

Guide for notation. In what follows,
ج denotes Dyers' outer automorphism of $\mathrm{PGL}_2(\mathbf{Z})$,
$\text{ج}_{\mathcal{F}}$ denotes the automorphism of the tree $|\mathcal{F}|$ induced by ج,
$\text{ج}_{\partial\mathcal{F}}$ denotes the homeomorphism of the boundary $\partial\mathcal{F}$ induced by $\text{ج}_{\mathcal{F}}$,

[1] This is the fifth letter of the arabic alphabet in the hijâ'î order. Latex preamble commands for a latin-compatible typography of ج are given at the end of the paper.

$\text{ج}_{\mathbf{R}}$ denotes the involution of $\mathbf{R}$ induced by $\text{ج}_{\partial\mathcal{F}}$, and
$\text{ج}_{\mathbf{Q}}$ denotes the involution of $\mathbf{Q}\setminus\{0,\infty\}$ induced by $\text{ج}_{\mathcal{F}}$.
However, we reserve the right to drop the subscript and simply write ج when we think that confusion won't arise.

Functional equations

The involution $\text{ج}_{\mathbf{R}}$ shares the privileged status of the fundamental involutions generating the extended modular group $\mathrm{PGL}_2(\mathbf{Z})$,

$$K : x \to 1-x, \quad U : x \to 1/x, \quad V : x \to -x,$$

as it interacts in a very harmonious way with them. Indeed, if we ignore its values at rational points then $\text{ج}_{\mathbf{R}}$ satisfies the following set of functional equations:

$$\text{(FE:0)} \quad \text{ج}(\text{ج}(x)) = x$$

$$\text{(FE:I)} \quad \text{ج}U = U\text{ج} \iff \text{ج}\left(\frac{1}{x}\right) = \frac{1}{\text{ج}(x)}$$

$$\text{(FE:II)} \quad \text{ج}V = UV\text{ج} \iff \text{ج}(-x) = -\frac{1}{\text{ج}(x)}$$

$$\text{(FE:III)} \quad \text{ج}K = K\text{ج} \iff \text{ج}(1-x) = 1-\text{ج}(x)$$

The following set of functional equations are derived from the above ones:

$$\text{(FE:II}'\text{)} \quad \text{ج}UV = V\text{ج} \iff \text{ج}\left(-\frac{1}{x}\right) = -\text{ج}(x)$$

$$\text{(FE:IV)} \quad \text{ج}KV = KUV\text{ج} \iff \text{ج}(1+x) = 1+\frac{1}{\text{ج}(x)}$$

$$\text{(FE:V)} \quad (\text{ج}_{\mathbf{R}}M\text{ج}_{\mathbf{R}})(x) = (\text{ج}M)(x) \quad \forall M \in \mathrm{PGL}_2(\mathbf{Z})$$

$$\text{(FE:VI)} \quad \text{ج}_{\mathbf{R}}(Mx) = \text{ج}M\text{ج}_{\mathbf{R}}(x) \quad \forall M \in \mathrm{PGL}_2(\mathbf{Z})$$

Equations (FE:I) and (FE:III) states that $\mathfrak{Z}$ is covariant with the operators U and K, whereas equation (FE:II) is a kind of lax-covariance Since U, V and K generate the group $\mathrm{PGL}_2(\mathbf{Z})$, we may say that $\mathfrak{Z}$ is lax-covariant with the $\mathrm{PGL}_2(\mathbf{Z})$-action on $\mathbf{R}$. Of course, if we consider $\mathfrak{Z}$ as an operator then covariance is simply a relation of commutativity. Relation (FE:IV) is not independent from the rest as it can be easily deduced from (FE:II) and (FE:III). The final relation is the most general form of the functional equations and says that $\mathfrak{Z}_{\mathbf{R}}$ conjugates the Möbius transformation $M \in \mathrm{PGL}_2(\mathbf{Z})$ to the Möbius transformation $\mathfrak{Z}(M) \in \mathrm{PGL}_2(\mathbf{Z})$. It is readily deduced from (FE:I-III) by using the involutivity of $\mathfrak{Z}$. As an instance of (FE:V), $\mathfrak{Z}$ conjugates the translation $1+x$ to $1+1/x$ and the transformation $1/(1+x)$ to $x/(1+x)$. (FE:VI) is got from (FE:V) by setting $x \to \mathfrak{Z}_{\mathbf{R}}(x)$. It says that $\mathfrak{Z}_{\mathbf{R}}$ is a kind of modular function.

Before attempting to play with them, beware that these functional equations are not consistent on the set of rational numbers. The first thing to try are the noble numbers. They are sent to rationals under $\mathfrak{Z}$ and $\mathfrak{Z}$ is 2-to-1 on this set. On the other hand, there is a related involution $\mathfrak{Z}_{\mathbf{Q}}$ on the set of positive rationals satisfying some functional equations. It must be stressed that $\mathfrak{Z}_{\mathbf{Q}}$ is not the restriction of $\mathfrak{Z}_{\mathbf{R}}$ to $\mathbf{Q}$. See page 28.

The privileged status of $\mathfrak{Z}$ is vindicated by the fact that it preserves the "real-multiplication locus", i.e. the set of real quadratic irrationalities. It does so in a highly non-trivial manner, though it preserves setwise the $\mathrm{PGL}_2(\mathbf{Z})$-orbits of real quadratic irrationalities. More generally $\mathfrak{Z}$ preserves setwise the $\mathrm{PGL}_2(\mathbf{Z})$-orbits on $\widehat{\mathbf{R}}$ thereby inducing an involution of the moduli space of real rank-2 lattices, $\widehat{\mathbf{R}}/\mathrm{PGL}_2(\mathbf{Z})$. Moduli of the lattices represented by the numbers $[\overline{1_k, k+2}]$ are fixed under this involution. For a precise description of all fixed points, see Proposition 22.

One could use the functional equations (FE:I)-(FE:III) to directly define and study the involution $\mathfrak{Z}_{\mathbf{R}}$. However, the elusive nature of $\mathfrak{Z}_{\mathbf{R}}$ is best understood by considering it as a homeomorphism of the boundary of the Farey tree, induced by an automorphism of the tree. This automorphism is the one which twists all but one vertex of the tree.

Finally, the following two-variable consequence of the functional equations is noteworthy:

$$\boxed{\frac{1}{x}+\frac{1}{y}=1 \iff \frac{1}{\mathfrak{Z}(x)}+\frac{1}{\mathfrak{Z}(y)}=1}$$

Hence $\mathfrak{Z}$ sends harmonic pairs of numbers to harmonic pairs. See page 29 for a complete set of two-variable functional equations.

Recall that, if the pair (x, y) satisfies $1/x+1/y=1$, then the so-called Beatty sequences $(\lfloor nx \rfloor)_{n\geq 1}$, $(\lfloor ny \rfloor)_{n\geq 1}$ gives a partition of the set of positive integers

(Rayleigh's theorem). Therefore every Beatty partition of positive integers admits a ζ-dual partition $(\lfloor n\zeta(x)\rfloor)_{n\geq 1}$, $(\lfloor n\zeta(y)\rfloor)_{n\geq 1}$.

Some examples.

Here is a list of assorted values of ζ. Recall that Φ denotes the golden section.

$$\Phi = [\overline{1}] \implies \zeta(\Phi) = \infty = \zeta\big(-1/\Phi\big), \tag{2}$$

where by $\overline{v}$ we denote the infinite sequence $v, v, v, \ldots$, for any finite sequence v. From 2 by using (FE:II) we find

$$\zeta(-\Phi) = \zeta\big(1/\Phi\big) = 0. \tag{3}$$

Repeated application of (FE:IV) gives

$$\zeta(n+\Phi) = \zeta([n+1, \overline{1}]) = F_{n+1}/F_n \tag{4}$$

where F_n denotes the nth Fibonacci number. One has

$$\zeta([1_n, 2, \overline{1}]) = [n+1, \infty] = n+1$$

For the number $\sqrt{2}$ we have something that looks simple

$$\sqrt{2} = [1, \overline{2}] \implies \zeta(\sqrt{2}) = [\overline{2}] = 1 + \sqrt{2}$$

but this is not typical as the next example illustrates:

$$\zeta((3+5\sqrt{2})/7) = \zeta([1, \overline{2, 3, 1, 1, 2, 1, 1, 1}]) = [2, \overline{2, 1, 4, 5}] = \frac{-3+2\sqrt{95}}{7}$$

In a similar vein, consider the examples

$$\sqrt{11} = [3; \overline{3, 6}] \implies \zeta(\sqrt{11}) = [\overline{1, 1, 2, 1, 2, 1, 1}] = \frac{15+\sqrt{901}}{26},$$

$$\zeta(-\sqrt{11}) = -1/\zeta(\sqrt{11}) = -\frac{26}{15+\sqrt{901}} = \frac{15-\sqrt{901}}{26}.$$

The last example hints at the following result

Theorem 1 *The involution ζ commutes with the conjugation of real quadratic irrationals; i.e. for every real quadratic irrational α one has $\zeta(\alpha^*) = \zeta(\alpha)^*$.*

Proof. Every real quadratic irrational α is the fixed point of some $M \in \mathrm{PSL}_2(\mathbf{Z})$, the Galois conjugate α^* being the other root of the equation $Mx = x$. But then $\text{ج}(M\alpha) = \text{ج}(M)\text{ج}(\alpha) = \text{ج}(\alpha)$, i.e. $\text{ج}(\alpha)$ is a fixed point of the equation $\text{ج}(M)$, the other root being $\text{ج}(\alpha)^*$. Finally

$$M\alpha = \alpha \implies M\alpha^* = \alpha^* \implies \text{ج}(\alpha^*) = \text{ج}(M\alpha^*) = \text{ج}(M)\text{ج}(\alpha^*),$$

so $\text{ج}(\alpha^*)$ is also a fixed point of $\text{ج}(M)$, i.e. it must coincide with $\text{ج}(\alpha)^*$. $\square$.

To finish, let us give the ج-transform of a non-quadratic algebraic number,

$$\begin{aligned} \text{ج}(\sqrt[3]{2}) &= \text{ج}([1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1, 3, 4, 1, \dots]) \\ &= [2, 1, 3, 1, 1, 1, 4, 1, 1, 4, 1_6, 3, 1_{12}, 3, 1_8, 2, 3, 1, 1, 2, 1_{10}, 2, 2, 1, 2, 3, 1, 2, 1, 1, 3, \dots] \\ &= 2.784731558662723\dots \end{aligned}$$

and the transforms of two familiar transcendental numbers:

$$\begin{aligned} \text{ج}(\pi) = \text{ج}([3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \dots]) &= [1_2, 2, 1_5, 2, 1_{13}, 3, 1_{290}, 5, 3, \dots] \\ &= 1.7237707925480276079699326494931025145558144289232\dots \end{aligned}$$

$$\begin{aligned} \text{ج}(e) = \text{ج}([2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]) &= [1, 3, 4, 1, 1, 4, 1, 1, 1, 1, \dots, \overline{4, 1_{2n}}] \\ &= 1.3105752928466255215822495496939143349712038085627\dots \end{aligned}$$

We have been unable to relate these numbers to other numbers of mathematics.

Algebraicity and transcendence

We made some numerical experiments on the algebraicity of a few numbers $\text{ج}(x)$ where x is an algebraic number of degree > 2, with a special emphasis on $\text{ج}(\sqrt[3]{2})$ (see our forthcoming paper). It is very likely that these numbers are transcendental. Furthermore, as we prove in Lemma 24, if X is a uniformly distributed random variable over the unit interval, then almost everywhere the average of continued fraction entries of $\text{ج}(X)$ tends to 1, i.e. $\text{ج}(X)$ does not obey the Gauss-Kuzmin statistics. As it is widely believed and experimentally affirmed that the algebraic numbers of degree≥ 3 do obey the Gauss-Kuzmin statistics, one can state with confidence the following conjecture:

Conjecture. If $x \in \mathbf{R}$ is algebraic of degree > 2, then $\text{ج}(x)$ is transcendental.

One philosophy concerning the continued fraction expansions of algebraic numbers is that it should not be possible to approximate them too closely by the rationals. On the other hand, since by the above remark on continued fraction entries, the convergents of the Ꞓ-transform of a uniformly distributed number converges almost surely very slowly, they can not be too closely approximated by the rationals, and this seem to provide some (rather weak) evidence against the conjecture.

It might be possible to tackle this conjecture with the methods recently introduced by Adamczewski and Y. Bugeaud, see [1].

Jimm-conjugates

Equation (FE:V) states that Ꞓ conjugates elements of $\mathrm{PGL}_2(\mathbf{Z})$ to elements of $\mathrm{PGL}_2(\mathbf{Z})$. In the next section we will see that Ꞓ conjugates the Gauss continued fraction map to something that makes sense. What about the other familiar functions of analysis? This question is probably not just a dull curiosity.

Conjecture. If M is an element of $\mathrm{PGL}_2(\mathbf{Q}) \setminus \mathrm{PGL}_2(\mathbf{Z})$, then the conjugation Ꞓ$M$Ꞓ assumes transcendental values at algebraic arguments of degree at least 3. Here $\mathrm{PGL}_2(\mathbf{Q})$ denotes the group of projective two-by-two *non-singular* integral matrices.

Dynamics

Denote by $\mathbb{T}_G$ the Gauss map $[0,1] \to [0,1]$, sending x to the fractional part of $1/x$ and denote by $\mathbb{T}_F$ the Farey map $[0,1] \to [0,1]$, defined as

$$\mathbb{T}_F(x) = \begin{cases} \frac{x}{1-x}, & 0 \le x \le 1/2 \\ \frac{1-x}{x}, & 1/2 < x \le 1 \end{cases} \tag{5}$$

The involution Ꞓ sends the unit interval onto itself, and the Ꞓ-conjugate of $\mathbb{T}_F$ is $\mathbb{T}_F$ itself with the two branches being permuted. On the other hand, the involution Ꞓ conjugates $\mathbb{T}_G$ (but not topologically) to a self-map $\mathbb{T}_{Ꞓ}$ of the unit interval, defined by

$$\mathbb{T}_{Ꞓ} : [0, 1_k, n_{k+1}, n_{k+2}, \dots] \in [0,1] \to [0, n_{k+1}-1, n_{k+2}, \dots] \in [0,1],$$

where it is assumed that $n_{k+1} > 1$ and $0 \le k < \infty$. (This map appears also in two recent papers [7], [22], where the name "Fibonacci map" was coined.) We discovered by trial and error that the resulting dynamical system have the infinite invariant measure $1/x(x+1)$, which can be readily verified The map $\mathbb{T}_{Ꞓ}$ gives rise to a transfer operator (recall that F_n is the nth Fibonacci number)

$$(\mathscr{L}_s^{Ꞓ}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(F_{k+1}y + F_k)^{2s}} \psi\left(\frac{F_k y + F_{k-1}}{F_{k+1}y + F_k}\right) \tag{6}$$

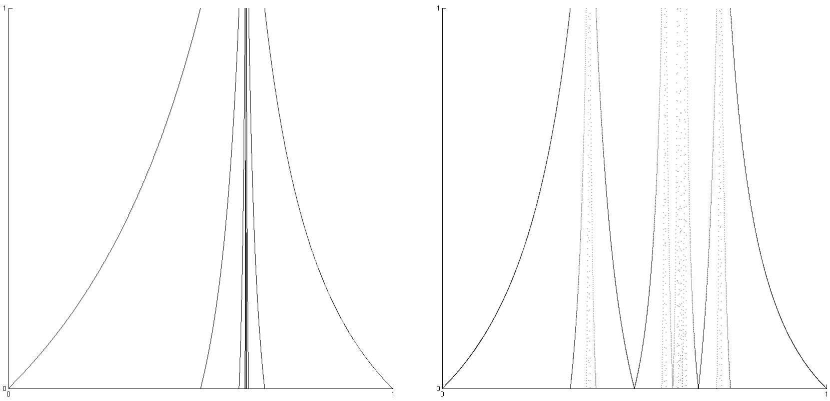


Figure 1: The graph of the map $\mathbb{T}_{jimm}$ and its first iterate.

eigenfunctions of which satisfies the three-term functional equation

$$\psi(y) = \frac{1}{y^{2s}}\psi\left(\frac{y+1}{y}\right) + \frac{1}{\lambda}\frac{1}{(y+1)^{2s}}\psi\left(\frac{y}{y+1}\right) \tag{7}$$

for the eigenvalue λ. It is straightforward to check that the solutions of this functional equation for $\lambda = 1$ corresponds in a one-to-one manner to solutions of Lewis' functional equations [54]),

$$\phi(y) = \phi(y+1) + \frac{1}{(y+1)^{2s}}\phi\left(\frac{1}{y+1}\right) \tag{8}$$

for the fixed points of the Mayer transfer operators [33]

$$(\mathscr{L}_s\phi)(y) = \sum_{k=1}^{\infty} \frac{1}{(y+n)^{2s}}\phi\left(\frac{1}{y+n}\right), \tag{9}$$

under the transformation

$$\phi(y) = y^{-2s}\psi(1/y) \leftrightarrow \psi(y) = y^{-2s}\phi(1/y). \tag{10}$$

The correspondence (10) works also for the fixed functions of the operators (9) and (6). Zagier and Lewis [31] established a correspondence between the Maass wave forms for the extended modular group and the solutions of the functional equation (8). Furthermore, according to a result of Mayer [33], the determinant of the operator $1 - \mathscr{L}_s$, when made to act on the space of holomorphic functions on the disc $\{|z-1| \leq 3/2\}$, exists in the Fredholm sense and equals the Selberg zeta function of $\mathrm{PGL}_2(\mathbf{Z})$. Hence, our story is related to the Selberg zeta, although we don't expect an exact statement of Mayer's result to hold for the operator $\mathscr{L}_s^{\mathfrak{C}}$. On the other hand, there is an alternative way of introducing some zeta analogues, by using the operator $\mathscr{L}_s^{\text{ج}}$. Since $\zeta_H(2s,x) = \mathscr{L}_s\mathbf{1}(x)$ is the Hurwitz zeta function, the function

$$\zeta_{\text{ج}}(2s,y) := \mathscr{L}_s^{\text{ج}}\mathbf{1}(y) = \sum_{k=1}^{\infty} \frac{1}{(F_{k+1}y+F_k)^{2s}}$$

arises as an analogue of the Hurwitz zeta and satisfies the functional equation

$$\zeta_{\text{ج}}(s,y) = y^{-s}\zeta_{\text{ج}}(s,1+1/y) + (y+1)^{-s}$$

It reduces to the so-called "Fibonacci zeta" when $y=0$

$$\zeta_{fib}(s) := \sum_{k=1}^{\infty} \frac{1}{F_k^s} = \zeta_{\text{ج}}(s,0),$$

studied for its own sake in the literature [35]. The $\zeta_{\text{ج}}$-values at $y=1$ and $y=2$ are also related to the Fibonacci zeta, via

$$\zeta_{\text{ج}}(s,1) = 2+\zeta_{fib}(s), \quad \zeta_{\text{ج}}(s,2) = 2+2^{-s}+\zeta_{fib}(s),$$

whereas $\zeta_{\text{ج}}$-value at $y=2$ is related to the so-called "Lucas zeta" [25]:

$$\zeta_{\text{ج}}(s,2) = 3^{-s}+\zeta_{luc}(s), \quad \zeta_{luc}(s) := \sum_{k=1}^{\infty} \frac{1}{(F_k+F_{k+2})^s}.$$

For more details about the dynamical system of $\mathbb{T}_{\text{ج}}$, its siblings and the associated zeta functions, see our forthcoming paper [48].

The involution ج is induced by an automorphism of the abstract Farey tree (denoted $|\mathcal{F}|$ in what follows). This connection calls for a systematic study of dynamical properties of the conjugates of the Gauss map by automorphisms of $|\mathcal{F}|$ that fix an edge I, denoted $Aut_I(|\mathcal{F}|)$. This latter group is an uncountable non-abelian profinite group (we give two quite concrete descriptions of its elements in this paper, in Theorems 2 and 3). In the same vein, it is of interest to know about the dynamical properties which distinguish $Aut_I(|\mathcal{F}|)$-orbits of dynamical maps on the unit interval.

It might also be of interest to study the ج-conjugates of other euclidean dynamical systems [50] (or of any other dynamical system with discontinuities at rationals for that matter) on $[0,1]$.

Combinatorics and arithmetic

Being an automorphism of $\mathrm{PGL}_2(\mathbf{Z})$, the automorphism ج acts on the system of subgroups $\mathbf{Sub}^*(\mathrm{PGL}_2(\mathbf{Z}))$ and on the system of conjugacy classes of subgroups

$\mathbf{Sub}(\mathrm{PGL}_2(\mathbf{Z}))$. We call the quotient $\mathrm{PGL}_2(\mathbf{Z})\backslash\mathcal{H}$ the *modular tile* and denote by $\mathcal{T}$. This is an orbifold with boundary and can also be described as the quotient of the modular curve $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathcal{H}$ under the complex conjugation. The action of ج on subgroups (respectively conjugacy classes of subgroups) induces an action on the base-pointed system of coverings $\mathbf{Cov}^*(\mathcal{T})$ (respectively the system of coverings $\mathbf{Cov}(\mathcal{T})$) of the modular tile. These coverings are surfaces possibly with boundary and with an orbifold structure. This action does not respect the genera of these surfaces. It does respect the rank of their fundamental groups. Jones and Thornton studied these actions to some depth in terms of the language of maps. One result they obtained (and besides Dyers', this is the only other result in the literature, about an action of ج that we are aware of) is

Theorem (Jones and Thornton [24]) *Let G be a congruence subgroup of* $\mathrm{PGL}_2(\mathbf{Z})$ *such that ج(G) is also a congruence subgroup. Then $G \geq \Gamma(600)$, where $\Gamma(N)$ the level-N principal congruence subgroup of* $\mathrm{PSL}_2(\mathbf{Z})$.

Since these covering systems are naturally equivalent to systems of (half-) ribbon graphs, we have the *combinatorial action* of ج on these graphs. This action can be related to an action on combinatorial objects called necklaces, bracelets, Lyndon words, to the indefinite integral binary quadratic forms of Gauss and also to an action on geodesics on the modular curve. This latter action does not respect nor preserve the geodesic length. It does preserve the primitivity. The ج-action on real quadratic irrationals studied in this paper to some depth, is related to these actions but we detail these connections elsewhere.

Numerics

There are many numerical excursions to be made around ج. One important challenge which we failed until now, is to recognize (i.e. express by some algebraic/analytic procedure other then via ج itself) the ج-transform of some number which is not rational and not a quadratic irrational. A second curiosity is to study the statistics of ج-transforms of some numbers (such as π or $\sqrt[3]{2}$) and see if they are "normal" in some proper sense. If you want to do some numerical experiments yourself, we plan to make our matlab and pyton codes and some graphics available via a tablet computer version of this paper and on our webpage.

2 Modular group and its automorphism group

The *modular group* is the projective group $\mathrm{PSL}_2(\mathbf{Z})$ of two by two unimodular integral matrices [29]. It acts on the upper half plane by Möbius transformations.

It is the free product of its subgroups generated by $S(z) = -1/z$ and $L(z) = (z-1)/z$, respectively of orders 2 and 3. Thus

$$\mathrm{PSL}_2(\mathbf{Z}) = \langle S, L \mid S^2 = L^3 = 1 \rangle$$

The projective group $\mathrm{PGL}_2(\mathbf{Z})$ consists of two by two integral matrices of determinant ± 1. Its Möbius action on the sphere does not preserve the upper half plane, i.e. if $\det(M) = -1$ then M exchanges the upper and lower half planes. There is a modification of this action which preserves the upper half plane, where $M \in \mathrm{PGL}_2(\mathbf{Z})$ acts as

$$M = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \rightsquigarrow \begin{cases} \frac{pz+q}{rz+s}, & if \quad \det(M) = 1 \\ \frac{p\bar{z}+q}{r\bar{z}+s}, & if \quad \det(M) = -1 \end{cases}$$

This representation of $\mathrm{PGL}_2(\mathbf{Z})$ is called the *extended modular group*. Here we are interested in the $\mathrm{PGL}_2(\mathbf{Z})$-action on the boundary circle of the upper half plane, so we do not need to distinguish between these actions. But we shall keep calling $\mathrm{PGL}_2(\mathbf{Z})$ "the extended modular group".

Note that, whenever we have an action of $\mathrm{PGL}_2(\mathbf{Z})$, we may compose it with ج and get another $\mathrm{PGL}_2(\mathbf{Z})$-action. Although it is essentially different from the first action, this latter action will have exactly the same invariants as the first one.

Below is a list of elements of $\mathrm{PGL}_2(\mathbf{Z})$ that will be used in the text.

Symbol	**Möbius map**	**Matrix form**
$S = UV$	$-1/x$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$L = KU$	$1 - 1/x$	$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$
V	$-x$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$T = LS = KV$	$x + 1$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
$\widetilde{T} = TU = KS$	$1 + 1/x$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
$U = SV$	$1/x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
K	$1 - x$	$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

The fact that $Out(\mathrm{PGL}_2(\mathbf{Z})) \simeq \mathbf{Z}/2\mathbf{Z}$ has a story with a twist: Hua and Reiner [19] determined the automorphism groups of various projective groups in 1952, claiming that $\mathrm{PGL}_2(\mathbf{Z})$ has no outer automorphisms. The error was corrected by

Dyer [14] in 1978. This automorphism also appears in the work of Djokovic and Miller [12] from about the same time. Dyer also proved that the automorphism tower of $\mathrm{PGL}_2(\mathbf{Z})$ stops here; i.e. $Aut(Aut(\mathrm{PGL}_2(\mathbf{Z}))) \simeq Aut(\mathrm{PGL}_2(\mathbf{Z}))$. Note that $\mathrm{PGL}_2(\mathbf{Z}) \simeq Aut(\mathrm{PSL}_2(\mathbf{Z}))$.

Below is a list of presentations of $\mathrm{PGL}_2(\mathbf{Z})$ in terms of several sets of generators, and the outer automorphism $\mathfrak{C}$ of $\mathrm{PGL}_2(\mathbf{Z})$ defined in terms of these generators.

Presentation of $\mathrm{PGL}_2(\mathbf{Z})$	**The automorphism** $\mathfrak{C}$
$\langle V, U, K \mid V^2 = U^2 = K^2 = (VU)^2 = (KU)^3 = 1\rangle$	$(V, U, K) \to (UV, U, K)$
$\langle V, U, L \mid V^2 = U^2 = (LU)^2 = (VU)^2 = L^3 = 1\rangle$	$(V, U, L) \to (UV, U, L)$
$\langle S, V, K \mid V^2 = S^2 = K^2 = (SV)^2 = (KSV)^3 = 1\rangle$	$(S, V, K) \to (V, S, K)$
$\langle T, U \mid U^2 = (T^{-2}UTU)^2 = (UTUT^{-1})^3 = 1\rangle$	$(T, U) \to (\widetilde{T}, U)$
$\langle T, \widetilde{T} \mid (T^{-1}\widetilde{T})^2 = (T^{-3}\widetilde{T}^2)^2 = (T^{-2}\widetilde{T}^2)^3 = 1\rangle$	$(T, \widetilde{T}) \to (\widetilde{T}, T)$

From the first presentation we see that Klein's viergruppe and the symmetric group on three letters are subgroups of $\mathrm{PGL}_2(\mathbf{Z})$:

$$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \simeq \langle V, U \mid V^2 = U^2 = 1\rangle < \mathrm{PGL}_2(\mathbf{Z}),$$

$$\Sigma_3 \simeq \langle U, K \mid K^2 = U^2 = (UK)^3 = 1\rangle < \mathrm{PGL}_2(\mathbf{Z}).$$

In fact $\mathrm{PGL}_2(\mathbf{Z})$ is an amalgamated product of these subgroups [11]:

$$\mathrm{PGL}_2(\mathbf{Z}) \simeq \langle V, U\rangle *_{\langle U\rangle} \langle V, K\rangle \simeq (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) *_{\mathbf{Z}/2\mathbf{Z}} \Sigma_3$$

In this description, we see that the outer automorphism of $\mathrm{PGL}_2(\mathbf{Z})$ originates from the following automorphism of Klein's viergruppe:

$$\mathfrak{C} : (I, U, V, UV) \to (I, U, UV, V)$$

We also see that $\mathfrak{C}$ leaves invariant the subgroup

$$\langle U, K\rangle \simeq \left\{ z, \frac{1}{z}, 1 - z, 1 - \frac{1}{z}, \frac{1}{1 - z}, \frac{z}{z - 1} \right\} \simeq \Sigma_3.$$

It is easy to find the $\mathfrak{C}$ of an element of $\mathrm{PGL}_2(\mathbf{Z})$ given as a word in one of the presentations listed above. On the other hand, there seems to be no algorithm to compute the $\mathfrak{C}$ of an element of $\mathrm{PGL}_2(\mathbf{Z})$ given in the matrix form, other then actually expressing the matrix in terms of one of the presentations above, finding the ج-transform, and then computing the matrix.

The trace $\text{ج}(M)$ has no simple expression in terms of the trace of M. It seems that $tr(\text{ج}(M))$ is a novel and subtle class invariant of $M \in \mathrm{PGL}_2(\mathbf{Z})$ and of the binary quadratic form obtained by homogenization from the fixed point equation of M.

It is easy to see that the translation $2 + x \in \mathrm{PSL}_2(\mathbf{Z})$ is sent to the transformation $(2x+1)/(x+1) \in \mathrm{PSL}_2(\mathbf{Z})$ under ج, i.e. it may send a parabolic element inside $\mathrm{PSL}_2(\mathbf{Z})$ to a hyperbolic element of inside $\mathrm{PSL}_2(\mathbf{Z}) \subset \mathrm{PGL}_2(\mathbf{Z})$. Here is a selecta of other examples illustrating the effect of ج on matrices:

M	$\text{ج}(M)$	M	$\text{ج}(M)$
$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}16 & 1\\ 15 & 1\end{bmatrix}$	$\begin{bmatrix}987 & 1597\\ 377 & 610\end{bmatrix}$
$\begin{bmatrix}15 & 1\\ 14 & 1\end{bmatrix}$	$\begin{bmatrix}610 & 987\\ 233 & 377\end{bmatrix}$	$\begin{bmatrix}29 & 2\\ 14 & 1\end{bmatrix}$	$\begin{bmatrix}843 & 1364\\ 610 & 987\end{bmatrix}$
$\begin{bmatrix}14 & 1\\ 13 & 1\end{bmatrix}$	$\begin{bmatrix}377 & 610\\ 144 & 233\end{bmatrix}$	$\begin{bmatrix}41 & 3\\ 27 & 2\end{bmatrix}$	$\begin{bmatrix}665 & 1076\\ 144 & 233\end{bmatrix}$
$\begin{bmatrix}27 & 2\\ 13 & 1\end{bmatrix}$	$\begin{bmatrix}521 & 843\\ 377 & 610\end{bmatrix}$	$\begin{bmatrix}40 & 3\\ 13 & 1\end{bmatrix}$	$\begin{bmatrix}898 & 1453\\ 521 & 843\end{bmatrix}$

Presentation of $Aut(\mathrm{PGL}_2(\mathbf{Z}))$

$\langle V, U, K, \text{ج} \,|\, V^2 = U^2 = K^2 = \text{ج}^2 = (VU)^2 = (KU)^3 = (\text{ج}V)^2U = (\text{ج}K)^2 = (\text{ج}U)^2 = 1\rangle$

One can eliminate $U = (\text{ج}V)^2$ from the presentation. Simplification gives

$$\langle V, K, \text{ج} \,|\, V^2 = K^2 = \text{ج}^2 = (\text{ج}V)^4 = (K\text{ج}V\text{ج}V)^3 = (\text{ج}K)^2 = 1\rangle$$

The largest Jimm-invariant subgroup. The subgroup $\mathrm{PSL}_2(\mathbf{Z})$ is not invariant under ج, its image is the subgroup

$$\text{ج}(\mathrm{PSL}_2(\mathbf{Z})) \simeq \langle V, L \,|\, V^2 = L^3 = 1\rangle \simeq \langle -z, 1 - 1/z\rangle$$

Note that ج sends elements of order three in $\mathrm{PSL}_2(\mathbf{Z})$ to elements of order three in $\mathrm{PSL}_2(\mathbf{Z})$: since the determinant is multiplicative, there are in fact no elements of order three in $\mathrm{PGL}_2(\mathbf{Z}) \setminus \mathrm{PSL}_2(\mathbf{Z})$.

The largest ج-invariant subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ is the index-2 subgroup

$$\mathrm{PSL}_2(\mathbf{Z}) \cap \text{ج}\mathrm{PSL}_2(\mathbf{Z}) = \langle R, SRS\rangle = \mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$$

This subgroup is the kernel

$$0 \longrightarrow \langle R, SRS\rangle \longrightarrow \mathrm{PSL}_2(\mathbf{Z}) \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0$$

Hence this is the subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ which consists of words in L and S with an even number of S's; in particular this subgroup does not contain any elliptic elements of order 2. Thus $\Gamma(2)$ and therefore all congruence subgroups $\Gamma(2N)$ are contained in it. On the other hand, since $T^{2N+1} \in \Gamma(2N+1)$, the subgroups $\Gamma(2N+1)$ are never contained in it.

3 The Farey tree and its boundary

3.1 The bipartite Farey tree $\mathcal{F}$.

(This subsection is taken to a large extent from a joint project of M. Uludağ with A. Zeytin on Thompson's groups.) From $\mathrm{PSL}_2(\mathbf{Z})$ we construct the *bipartite Farey tree* $\mathcal{F}$ tree on which $\mathrm{PSL}_2(\mathbf{Z})$ acts, as follows. The edges of $\mathcal{F}$ consists of the elements of the modular group; i.e. $E(\mathcal{F}) = \mathrm{PSL}_2(\mathbf{Z})$. The set of vertices of $\mathcal{F}$ are the left cosets of the subgroups $\langle S\rangle$ and $\langle L\rangle$. Two distinct vertices v and v' are joined by an edge if and only if the intersection $v \cap v'$ is non-empty and in this case the edge between the two vertices is the only element in the intersection. Since cosets of a subgroup are disjoint, $\mathcal{F}$ is a bipartite graph. Cosets of $\langle S\rangle$ are always 2-valent vertices and the cosets of $\langle L\rangle$ are always 3-valent. The edges incident to the vertex $\{W, WL, WL^2\}$ are W, WL and WL^2, and these edges inherit a natural cyclic ordering which we fix for all vertices as (W, WL, WL^2). This endows $\mathcal{F}$ with the structure of a ribbon graph. It is connected since $\mathrm{PSL}_2(\mathbf{Z})$ is generated by S and L and is circuit-free since it is freely generated by these elements. Hence $\mathcal{F}$ is an infinite bipartite tree with a ribbon structure.

$M \in \mathrm{PSL}_2(\mathbf{Z})$ acts on $\mathcal{F}$ from the left by ribbon graph automorphisms by sending the edge labeled W to the edge labeled MW. This action is free on $E(\mathcal{F})$ but not free on the set of vertices: the vertex $\{W, WL, WL^2\}$ is left invariant by the order-3 subgroup $\langle WLW^{-1}\rangle$ and the vertex $\{W, WS\}$ is left invariant by the order-2 subgroup $\langle WSW^{-1}\rangle$.

The boundary of $\mathcal{F}$. A *path* on a graph $\mathcal{G}$ is a sequence of edges $e_1, e_2, \ldots, e_k$ of $\mathcal{G}$ such that e_i and e_{i+1} meet at a vertex, for each $1 \leq i < k-1$. Since the edges of $\mathcal{F}$ are labeled by reduced words in the letters L and S, a path in $\mathcal{F}$ is a sequence of reduced words (W_i) in L and S, such that $W_i^{-1}W_{i+1} \in \{L, L^2, S\}$ for every i. Since $\mathcal{F}$ is a tree, there is a unique non-backtracking path through any two edges.

An *end* of $\mathcal{F}$ is an equivalence class of infinite (but not bi-infinite) non-backtracking paths in $\mathcal{F}$, where eventually coinciding paths are considered as equivalent. In other words, an end of $\mathcal{F}$ is the equivalence class of an infinite sequence of finite reduced words (W_i) in L and S with $W_i^{-1}W_{i+1} \in \{L, S\}$ for every i, where sequences with coinciding tails are equivalent.

The set of ends of $\mathcal{F}$ is denoted $\partial\mathcal{F}$. The action of $\mathrm{PSL}_2(\mathbf{Z})$ on $\mathcal{F}$ extends to an action on the set $\partial\mathcal{F}$, the element $M \in \mathrm{PSL}_2(\mathbf{Z})$ sending the path (W_i) to the path (MW_i).

Given an edge e of $\mathcal{F}$ and an end b of $\mathcal{F}$, there is a unique path in the class b which starts at e. Hence for any edge e, we may identify the set $\partial\mathcal{F}$ with the set of infinite non-backtracking paths that start at e. We denote this latter set by $\partial\mathcal{F}_e$ and endow it with the product topology. This topology is generated by the open sets, called *Farey intervals* $\mathcal{O}_{e'}$, which are defined to be the set of infinite paths starting at e and passing through e. The space $\partial\mathcal{F}_e$ is also endowed with a natural cyclic ordering induced by the ribbon structure of $\mathcal{F}$. Hence $\partial\mathcal{F}_e$ is a cyclically ordered topological space. Given a second edge e' of $\mathcal{F}$, the spaces $\partial\mathcal{F}_e$ and $\partial\mathcal{F}_{e'}$ are homeomorphic under the order-preserving map which pre-composes with the unique path joining e to e'. The action of the modular group on the set $\partial\mathcal{F}$ induces an action of $\mathrm{PSL}_2(\mathbf{Z})$ by order-preserving homeomorphisms of the topological space $\partial\mathcal{F}_e$, for any choice of a base edge e.

This construction of the Farey tree and that of its boundary below, was inspired by Qaiser Mushtaq's work [36], [37] on coset diagrams, and also by a paper of Manin and Marcolli [32]. It can be established for a large class of trees with a planar structure, see [40] and [39].

The continued fraction map. $\partial\mathcal{F}_e$ is homeomorphic to the Cantor set, i.e. it is an uncountable, compact, totally disconnected, Hausdorff topological space. By exploiting its cyclic order structure, we "smash the holes" of this Cantor set to obtain the continuum, as follows. Define a *rational end* of $\mathcal{F}$ to be an eventually left-turn or eventually right-turn path. Now introduce the equivalence relation $\sim_*$ on $\partial\mathcal{F}$ as: left- and right- rational paths which bifurcate from the same vertex are equivalent. On $\partial\mathcal{F}_e$ this equivalence sets equal those points which are not separated by (with respect to the order relation) a third point.

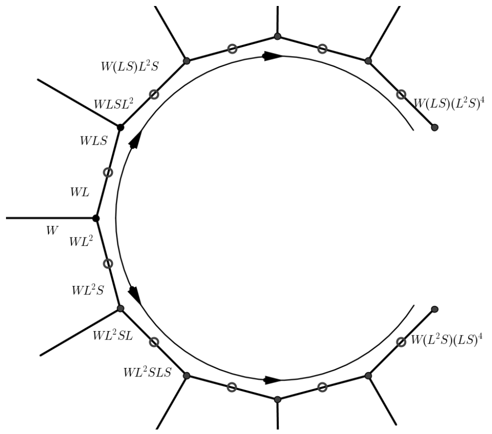


Figure. A pair of rational ends.

On the quotient space $\partial\mathcal{F}_e/\sim_*$ there is the quotient topology induced by the topology on $\partial\mathcal{F}_e$ such that the projection map

$$\partial\mathcal{F}_e \longrightarrow \partial\mathcal{F}_e/\sim_* \tag{11}$$

is continuous. The quotient space[2] $\partial\mathcal{F}_e/\sim_*$ is a cyclically ordered topological space under the order relation inherited from $\partial\mathcal{F}_e$. We shall denote this quotient space by S^1_e. The equivalence relation is preserved under the canonical homeomorphisms $\partial\mathcal{F}_e \longrightarrow \partial\mathcal{F}_{e'}$ and is also respected by the $\mathrm{PSL}_2(\mathbf{Z})$-action. Therefore we have the commutative diagram

$$\begin{array}{ccc} \partial\mathcal{F}_e & \longrightarrow & \partial\mathcal{F}_{e'} \\ \downarrow & & \downarrow \\ S^1_e & \longrightarrow & S^1_{e'} \end{array}$$

where the horizontal arrows are order-preserving homeomorphisms and the vertical arrows are projections. Moreover, $\mathrm{PSL}_2(\mathbf{Z})$ acts by homeomorphisms on S^1_e, for any e.

Now, $\mathcal{F}$ comes equipped with a distinguished edge, the one marked I, the identity element of the modular group. Hence all spaces S^1_e are canonically homeomorphic to S^1_I.

Any element of S^1_I can be represented by an infinite word in L and S. Regrouping occurrences of LS and L^2S, any such word x of S^1_I can be written in one of the following forms:

$$x = (LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\cdots \text{ or} \tag{12}$$

$$x = S(LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\cdots, \tag{13}$$

[2]The space $S^1_e := \partial\mathcal{F}_e/\sim_*$ is called the circle boundary of $\mathcal{F}$, Northshield [39] gave a much general construction for planar graphs, in the context of potential theory.

where $n_0, n_1 \cdots \geq 0$. Since our paths do not have any backtracking we have $n_0 \geq 0$ and $n_i > 0$ for $i = 1, 2, \cdots$. The pairs of words

$$(LS)^{n_0} \cdots (LS)^{n_k+1}(L^2S)^\infty \text{ and } (LS)^{n_0} \cdots (LS)^{n_k}(L^2S)(LS)^\infty, \qquad (k \text{ even})$$
$$(LS)^{n_0} \cdots (L^2S)^{n_k+1}(LS)^\infty \text{ and } (LS)^{n_0} \cdots (L^2S)^{n_k}(LS)(L^2S)^\infty, \qquad (k \text{ odd})$$

correspond to pairs of rational ends and represent the same element of S^1_I. For irrational ends this representation is unique.

The $\mathrm{PSL}_2(\mathbf{Z})$-action on S^1_I is then the pre-composition of the infinite word by the word in L, S representing the element of $\mathrm{PSL}_2(\mathbf{Z})$. In this picture it is readily seen that this action respects the equivalence relation $\sim_*$

Set $T := LS$, so that $T(x) = 1 + x$. Note that

$$\begin{aligned}(LS)^n.(L^2S)^m.(LS)^k(x) &= (LS)^n.S.[S.(L^2S)^m.S].S.(LS)^k(x)\\ = (LS)^n.S.[SL^2]^m.S.(LS)^k(x) &= (LS)^n.S.[LS]^{-m}.S.(LS)^k(x)\\ = (x+n)\circ(-1/x)\circ(x-m)\circ(-1/x)\circ(x+k) &= n + \cfrac{1}{m + \cfrac{1}{k+x}}\end{aligned}$$

Accordingly, define the *continued fraction map* $cfm : S^1_I \to \hat{\mathbf{R}}$ by

$$cfm(x) = \begin{cases} [n_0, n_1, n_2, \ldots] & \text{if } x = (LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\ldots \\ -1/[n_0, n_1, n_2, \ldots] & \text{if } x = S(LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\ldots \end{cases}$$

Theorem 2 *The continued fraction map cfm is a homeomorphism.*

Proof. To each rational end $[n_0, n_1, n_2, \ldots, n_k, \infty]$ we associate the rational number $[n_0, n_1, n_2, \ldots, n_k]$. Likewise for the rational ends in the negative sector. This is an order preserving bijection between the set of equivalent pairs of rational ends and $\mathbf{Q} \cup \{\infty\}$. Now observe that an infinite path is then no other than a Dedekind cut and conversely every cut determines a unique infinite path, see [49] for details. □

As a consequence of this result, we see that the continued fraction map conjugates the $\mathrm{PSL}_2(\mathbf{Z})$-action on S^1_I to its action on $\widehat{\mathbf{R}}$ by Möbius transformations. Furthermore, there is a bijection between $\widehat{\mathbf{Q}}$ and the set of pairs of equivalent rational ends. Any pair of equivalent rational ends determines a unique *rational horocycle*, a bi-infinite left-turning (or right-turning) path. Hence, there is a bijection between $\widehat{\mathbf{Q}}$ and the set of rational horocycles.

Lemma 3 *Given the base edge I of $\mathcal{F}$,*
(i) There is a natural bijection between the 3-valent vertices of $\mathcal{F}$ and $\widehat{\mathbf{Q}} \setminus \{0, \infty\}$.
(ii) There is a bijection between the 2-valent vertices and the Farey intervals $[p/q, r/s]$ with $ps - qr = 1$.

Proof. (i) On each rational horocycle there lies unique trivalent vertex which is closest to the base edge I. If we exclude the two horocycles on which I lies, this gives a bijection between the set of trivalent vertices of $\mathcal{F}$ and the set of horocycles. The continued fraction map sends the two horocycles through I to 0 and ∞. Hence, there is a natural correspondence between the set of trivalent vertices of $\mathcal{F}$ and $\widehat{\mathbf{Q}} \setminus \{0, \infty\}$. This correspondence sends the trivalent vertex $\{W, WL, WL^2\}$ to $W(1) \in \mathbf{Q}$, where it is assumed that W is a reduced word which ends with an S. (ii) Every 2-valent vertex $\{W, WS\}$ lies exactly on two horocycles, and the set of paths based at I and through $\{W, WS\}$ is sent to the interval $[W(0), WS(0)]$ under cfm. $\square$

Periodic paths and the real multiplication set. Let $\gamma := (W_1, W_2, \dots, W_n)$ be a finite path in $\mathcal{F}$. Then the *periodization* of γ is the path γ^ω defined as

$$W_1, W_2, \dots, W_n, W_n W_1^{-1} W_2, W_n W_1^{-1} W_3, \dots, W_n W_1^{-1} W_n,$$
$$(W_n W_1^{-1})^2 W_2, \dots, (W_n W_1^{-1})^2 W_n, (W_n W_1^{-1})^3 W_2, \dots$$

In plain words, γ^ω is the path obtained by concatenating an infinite number of copies of a path representing $W_1^{-1} W_n$, starting at the edge W_1. If $W_1^{-1} W_n$ is elliptic, then γ^ω is an infinitely backtracking finite path. If not, γ^ω is actually infinite and represents an end of $\mathcal{F}$. We call these *periodic ends of* $\mathcal{F}$. Thus we have the periodization map

$$per: \{\, finite\ hyperbolic\ paths \,\} \longrightarrow \{\, ends\ of\ \mathcal{F} \,\}$$

whose image consists of periodic ends. The map *per* is not one-to one but its restriction to the set of primitive paths is one-to-one. The modular group action on $\partial\mathcal{F}$ preserves the set of periodic ends and the periodization map is $\mathrm{PSL}_2(\mathbf{Z})$-equivariant.

Given an edge e of $\mathcal{F}$ and a periodic end b of $\mathcal{F}$, there is a unique path in the class b which starts at e. This way the set of periodic ends of $\mathcal{F}$ is identified with the set of eventually periodic paths based at e. This set is dense in $\partial\mathcal{F}_e$ and preserved under the canonical homeomorphisms between the spaces $\partial\mathcal{F}_e$ and $\partial\mathcal{F}_{e'}$. Every periodic end has a unique $\mathrm{PSL}_2(\mathbf{Z})$-translate, which is a purely periodic path based at e. Finally, the set of periodic ends descends to a well-defined subset of S^1_e. The image of this set under the continued fraction map consists of the set of eventually periodic continued fractions, i.e. the set of real quadratic irrationalities (the "real-multiplication set"). These are precisely the fixed points of the $\mathrm{PSL}_2(\mathbf{Z})$-action on S^1_e; a hyperbolic element $M = (LS)^{n_0}(L^2S)^{n_1} \cdots (LS)^{n_k} \in \mathrm{PSL}_2(\mathbf{Z})$ fixing the numbers represented by the infinite words

$$(LS)^{n_0}(L^2S)^{n_1} \cdots (LS)^{n_k}(LS)^{n_0}(L^2S)^{n_1} \cdots (LS)^{n_k} \dots, \text{ and}$$
$$(LS)^{-n_k} \cdots (L^2S)^{-n_1}(LS)^{-n_k}(LS)^{-n_k} \cdots (L^2S)^{-n_1}(LS)^{-n_k} \dots.$$

3.2 The automorphism group of $\mathcal{F}$

Recall that $\mathrm{PSL}_2(\mathbf{Z})$ acts on $\mathcal{F}$ by ribbon graph automorphisms and on $\partial\mathcal{F}$ by homeomorphisms respecting the pairs of rational ends thereby acting by homeomorphisms on S^1. Any non-identity element of $\mathrm{PSL}_2(\mathbf{Z})$ is either of finite order, stabilizes a pair of rational ends, or it stabilizes a pair of eventually periodic ends of $\mathcal{F}$; in which case it is respectively called *elliptic*, *parabolic* or *hyperbolic*.

Now let us forget about the ribbon structure of $\mathcal{F}$. This gives an abstract graph which we denote by $|\mathcal{F}|$. The automorphism group[3] $Aut(|\mathcal{F}|)$ of $|\mathcal{F}|$ is much bigger then $Aut(\mathcal{F})$. It is uncountable and contains $Aut(\mathcal{F})$ as a non-normal subgroup. It is not compact but locally compact under its natural topology. In this topology, a neighborhood base of an automorphism γ consists those elements of $Aut(|\mathcal{F}|)$ which agree with γ on finite subtrees.

The map

$$Aut(|\mathcal{F}|) \times \partial\mathcal{F} \to \partial\mathcal{F}$$

is continuous and $Aut(|\mathcal{F}|)$ also acts by homeomorphisms on $\partial\mathcal{F}$. However, automorphisms of $|\mathcal{F}|$ does not respect the ribbon structure on $\mathcal{F}$ in general and does not induce a well-defined homeomorphism of S^1_I in general (for example, as in the case of $\mathfrak{C}_{\partial\mathcal{F}}$, an automorphism of $|\mathcal{F}|$ may send a pair of rational ends to distinct irrational ends.).

A bi-infinite path without back-tracking in $\mathcal{F}$ is called an *oriented geodesic* of $\mathcal{F}$; the set of geodesics of $\mathcal{F}$ is in one-to-one correspondence with the "Cantor torus" $\partial\mathcal{F} \times \partial\mathcal{F}$, minus the diagonal. In other words, every pair of distinct ends determine a unique oriented geodesic.

Fix an edge e of $\mathcal{F}$ and denote by $Aut_e(|\mathcal{F}|)$ the group of automorphisms of $|\mathcal{F}|$ that stabilize e. For any pair e, e' of edges, $Aut_e(|\mathcal{F}|)$ and $Aut_{e'}(|\mathcal{F}|)$ are conjugate subgroups.

For $n > 0$ let $|\mathcal{F}_n|$ be the finite subtree of $|\mathcal{F}|$ containing vertices of distance $\leq n$ from e. Then $|\mathcal{F}_n| \subset |\mathcal{F}_{n+1}|$ forms an injective system with respect to inclusion and $Aut_e(|\mathcal{F}_{n+1}|) \to Aut_e(|\mathcal{F}_n|)$ forms a projective system, and one has

$$Aut_e(|\mathcal{F}|) = \varprojlim Aut_e(|\mathcal{F}_n|).$$

Hence $Aut_e(|\mathcal{F}|)$ is a profinite group. Note that any edge e splits $\mathcal{F}$ into two components and each one of these components are preserved by all elements of $Aut_e(|\mathcal{F}|)$. Hence,

$$Aut_e(|\mathcal{F}|) \simeq Aut(\mathcal{T}) \times Aut(\mathcal{T}),$$

[3]The group $Aut(|\mathcal{F}|)$ is naturally isomorphic to the automorphism group of the abstract trivalent tree obtained from $|\mathcal{F}|$ by forgetting the vertices of degree 2.

where $\mathcal{T}$ is the rooted infinite binary tree. The group $Aut(\mathcal{T})$ might be described as a certain wreath product of an infinite number of copies of $\mathbf{Z}/2\mathbf{Z}$ (see Nekrashevych [38]), but we prefer and we shall present below two alternative descriptions which appears to be more intuitive.

A description of tree automorphisms as shuffles. Let us turn back to the ribbon graph $\mathcal{F}$, which by construction comes with a base edge, the one labeled with $I \in \mathrm{PSL}_2(\mathbf{Z})$. Given any vertex v, this base edge permits us to speak about the full subtree (i.e. Farey branch) attached to $\mathcal{F}$ at v.

Denote by $V_\bullet$ the set of vertices of degree three of $\mathcal{F}$.

The ribbon structure of $\mathcal{F}$ serves as a sort of coordinate system to describe all automorphisms of $|\mathcal{F}|$, as follows.

Given a vertex $v = \{W, WL, WL^2\}$ of type $V_\bullet$ of $\mathcal{F}$, the *shuffle* σ_v is the automorphism of $|\mathcal{F}|$ which is defined as:

$$\sigma_v : \text{edge labeled } M \longrightarrow \text{edge labeled} \begin{cases} M, & \text{if } M \neq WLX \\ WL^2X, & \text{if } M = WLX \\ WLX, & \text{if } M = WL^2X \end{cases} \tag{14}$$

where M and X are assumed to be reduced words in S and L. Thus σ_v is the identity away from the Farey branches at v, whereas it exchanges the two Farey branches at v. Note that $\sigma_v^2 = I$, i.e. the shuffle σ_v is involutive.

The automorphism σ_v is obtained by permuting the two branches attached at v, by shuffling these branches one above the other. Beware that σ_v is *not* the automorphism θ_v of $|\mathcal{F}|$ obtained by rotating in the physical 3-space the branches starting at WL and at WL^2 around the vertex v. We call this latter automorphism a *twist*, see below for a precise definition.

We must also stress that the definition of σ_v requires the ribbon structure of $\mathcal{F}$ as well as a base edge, although σ_v is never an automorphism of $\mathcal{F}$.

Evidently, σ_v stabilizes I, so one has $\sigma_v \in Aut_I(|\mathcal{F}|)$.

Given an arbitrary (finite or infinite) set ν of vertices in $V_\bullet$, we inductively define the shuffle $\sigma_\nu \in Aut_I(|\mathcal{F}|)$ as follows: First order the elements of ν with respect to the distance from the base edge I, i.e. set $\nu^{(1)} = (v_1^{(1)}, v_2^{(1)}, \dots)$, where $v_i^{(1)} \neq v_{i+1}^{(1)}$ and such that

$$d(v_i^{(1)}, I) \leq d(v_{i+1}^{(1)}, I) \text{ for all } i = 1, 2, \dots$$

For $i = 1, 2 \dots$ set

$$\sigma_\nu^{(i)} := \sigma_{v_i^{(i)}} \text{ and } v_j^{(i+1)} = \sigma_\nu^{(i)}(v_j^{(i)}) \text{ for } j \geq i+1$$

Since for any $j > i$, the automorphism $\sigma_\nu^{(j)}$ agrees with $\sigma_\nu^{(i)}$ on $\mathcal{F}_i$, the sequence $\sigma_\nu^{(i)}$ converges in $Aut_I(|\mathcal{F}|)$ and we set

$$\sigma_\nu := \lim_{i\to\infty} \sigma_\nu^{(i)}$$

for the limit automorphism of $|\mathcal{F}|$. There is some arbitrariness in the initial ordering of ν, concerning its elements of constant distance to the edge I, but the limit does not depend on this. The reason is that the shuffles corresponding to those elements commute. $\sigma_\emptyset$ is the identity automorphism by definition. Note that σ_ν is not involutive in general.

This gives us a unique opportunity to glimpse inside an infinite, non-abelian profinite group; any automorphism of $|\mathcal{F}|$ that fix the edge I is in fact a shuffle:

Theorem 4 $Aut_I(|\mathcal{F}|) = \{\sigma_\nu \mid \nu \subseteq V_\bullet(\mathcal{F})\}$.

Proof. It suffices to show that the restrictions of shuffles to finite subtrees $|\mathcal{F}_n|$ gives the full group $Aut_I(|\mathcal{F}_n|)$. This is easy. □

Beware the trade-off: in the shuffle description, elements of $Aut_I(|\mathcal{F}|)$ are quite visible whereas the group operation is not so direct, as attempts to compute some powers of some non-trivial elements shows.

The automorphism $\sigma_{V_\bullet}$. What happens if we shuffle every $\bullet$-vertex of $|\mathcal{F}|$? In other words, what is effect of the automorphism $\sigma_{V_\bullet}$ on $\partial_I F$? If $x \in \partial_I F$ is represented by an infinite word[4] in L, L^2 and S, then according to (14), we see that $\sigma_{V_\bullet}$ replaces L by L^2 and vice versa. In other words, if x is of the form

$$x = (LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\ldots,$$

then one has

$$\sigma_{V_\bullet}(x) = (L^2S)^{n_0}(LS)^{n_1}(L^2S)^{n_2}(LS)^{n_3}(L^2S)^{n_4}\ldots$$

In terms of the contined fraction representations, we get, for $n_0 > 0$,

$$\sigma_{V_\bullet}([n_0, n_1, n_2, \ldots]) = [0, n_1, n_2, \ldots]$$

and for $n_0 = 0$

$$\sigma_{V_\bullet}([0, n_1, n_2, \ldots]) = [n_1, n_2, \ldots].$$

[4]In what follows, we will be sloppy about the difference between a path $x \in \partial_I\mathcal{F}$, its equivalence class in S^1_I, and the real number it represents, i.e. its image $cfm(x)$ under the continued fraction map.

It is readily verified that a similar formula also holds when x starts with an S, and we get

$$\sigma_{V_\bullet}(x) = 1/x = U(x).$$

Observe (keeping in mind the forthcoming parallelism between ج and U) that U is an involution satisfying the equations

$$U = U^{-1}, \quad US = SU, \quad LU = UL^2. \tag{15}$$

Here, U, L and S are viewed as operators acting on the boundary $\partial_I \mathcal{F}$. These equations can be re-written in the form of functional equations as below:

$$U(Ux) = x, \quad U(-\frac{1}{x}) = -\frac{1}{Ux}, \quad U(\frac{1}{1-x}) = 1 - \frac{1}{Ux}.$$

One may derive other functional equations from these, i.e. $UT = ULS = L^2US = L^2SU$ is written as

$$U(1+x) = \frac{Ux}{1+Ux}.$$

One may consider U as an "orientation-reversing" automorphism of the ribbon tree $\mathcal{F}$. In the same vein, U is a homeomorphism of $\partial_I \mathcal{F}$ which reverses its canonical ordering. It is the sole element of $Aut_I(|\mathcal{F}|)$ which respects the equivalence $\sim_*$ (unlike ج) and hence U acts by homeomorphism on S^1_I.

In fact, taken as a map on $E(\mathcal{F}) = \mathrm{PSL}_2(\mathbf{Z})$, the involution U is an automorphism of $\mathrm{PSL}_2(\mathbf{Z})$, the one defined by

$$U : \begin{pmatrix} S \\ L \end{pmatrix} \to \begin{pmatrix} S \\ L^2 \end{pmatrix}$$

The automorphism group of $\mathrm{PSL}_2(\mathbf{Z})$ is generated by the inner automorphisms and U: this is the group $\mathrm{PGL}_2(\mathbf{Z})$.

Twists. Let $v \in V_\bullet$ and let ν_v be the set of all vertices (including v) on the Farey branch (with reference to the edge I) of $\mathcal{F}$ at v. Then the *twist* of v is the automorphism of $\mathcal{F}$ defined by

$$\theta_v := \sigma_{\nu_v}.$$

In words, θ_v is the shuffle of every vertex of the Farey branch at v. As in the case of shuffles, for any subset $\mu \subset V_\bullet$, one may define the twist θ_μ as a limit of convergent sequence of individual twists.

Lemma 5 *Let v be a vertex in $V_\bullet$ and v', v'' its two children (with respect to the ancestor I). Put $\mu := \{v, v', v''\}$. Then $\sigma_v = \theta_\mu$.*

Hence, shuffles can be expressed in terms of twists and vice versa. This proves the following result.

Theorem 6 $Aut_I(|\mathcal{F}|) = \{\theta_\mu \,|\, \mu \subseteq V_\bullet(\mathcal{F})\}$.

DEFINITION-NOTATION 7 *Let v^* be the trivalent vertex incident to the base edge I, i.e. $v^* := \{I, L, L^2\}$ and set $V_\bullet^* := V_\bullet \backslash \{v^*\}$. We denote the special automorphism $\theta_{V_\bullet^*}$ by $\mathfrak{Z}_{\mathcal{F}}$. In words, this is the automorphism of $|\mathcal{F}|$ obtained by twisting every trivalen vertex except the vertex v^*. This is the same automorphism obtained by shuffling every other trivalent vertex, such that the vertex v^* is not shuffled.*

More precisely, the following consequence of Lemma 5 holds:

Lemma 8 *One has $\theta_{V_\bullet^*} = \mathfrak{Z}_{\mathcal{F}} = \sigma_J$, where $J \subset V_\bullet$ is the set of degree-3 vertices, whose distance to the base edge is an odd number.*

Obviously, $\mathfrak{Z}_{\mathcal{F}}$ induce an involutive homeomorphism of $\partial_I\mathcal{F}$ which do not respect its canonical ordering. In fact, one may say that it destroys the ordering of the boundary in the most terrible possible way. We denote this homeomorphism by $\mathfrak{Z}_{\partial\mathcal{F}}$. Terrible as they are, we must emphasize that $\mathfrak{Z}_{\mathcal{F}}$ and $\mathfrak{Z}_{\partial\mathcal{F}}$ are perfectly well-defined mappings on their domain of definition. They don't exhibit such things as the two-valued behavior of $\mathfrak{Z}_{\mathbf{R}}$ at rationals. This two-valued behavior is a consequence of the fact that $\mathfrak{Z}_{\partial\mathcal{F}}$ do not respect the equivalence relation $\sim_*$.

To see the effect of $\mathfrak{Z}_{\partial\mathcal{F}}$ on $x \in \partial_I\mathcal{F}$, assume

$$x = S^\epsilon(LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\ldots, \; (n_0 \geq 0,\, n_i > 0 \text{ if } i > 0,\, \epsilon \in \{0,1\}).$$

We may represent this element by a string of 0's and 1's (0 for L and 1 for L^2):

$$x = S^\epsilon \underbrace{00\ldots0}_{n_0}\underbrace{11\ldots1}_{n_1}\underbrace{00\ldots0}_{n_2}\underbrace{11\ldots1}_{n_3}\ldots,$$

Then rational numbers are represented by the eventually constant strings where for any finite string a, the strings $a0111\ldots$ and $a1000\ldots$ represent the same rational number[5].

Let $\phi := (01)^\omega$ be the zig-zag path to infinity, and set $\phi^* := \neg\phi = (10)^\omega$. Then

$$\mathfrak{Z}_{\partial\mathcal{F}}(x) = \begin{cases} a\underline{\vee}\phi, & x = a \in \{0,1\}^\omega, \\ S(a\underline{\vee}\phi*), & x = Sa,\, a \in \{0,1\}^\omega. \end{cases}$$

[5]The two representations of the number $0 \in \mathbf{R}$ are 1^ω and -0^ω and the two representations of $\infty \in \widehat{\mathbf{R}}$ are 0^ω and -1^ω.

where $\veebar$ is the operation of term-wise exclusive or (XOR) on the strings of 0's and 1's. The involutivity of $\mathrm{J}_{\partial\mathcal{F}}$ then stems from the reversibility of the disjunctive or: $(p \veebar q) \veebar q = p$.

Some Examples. The string $0(0011)^\omega$ corresponds to the continued fraction $[1, 2, 2, 2, \dots]$, which equals $\sqrt{2}$. One has

$\partial\mathcal{F}$	$\{0,1\}^\omega$	$\widehat{\mathbf{R}}$
x	0 1 1 0 0 1 1 0 0 1 1 0...	$\sqrt{2}$
ϕ	0 1 0 1 0 1 0 1 0 1 0 1...	Φ
$\mathrm{J}(x)$	0 0 1 1 0 0 1 1 0 0 1 1...	$1+\sqrt{2}$

As for the value of $\mathrm{J}_{\mathbf{R}}$ at ∞, one has

$\partial\mathcal{F}$	$\{0,1\}^\omega$	$\widehat{\mathbf{R}}$
x	0 0 0 0 0 0 0 0 0 0 0 0...	∞
ϕ	0 1 0 1 0 1 0 1 0 1 0 1...	Φ
$\mathrm{J}(x)$	0 1 0 1 0 1 0 1 0 1 0 1...	Φ

The two values $\mathrm{J}_{\mathbf{R}}$ assumes at the point 1 are found as follows:

$\partial\mathcal{F}$	$\{0,1\}^\omega$	$\widehat{\mathbf{R}}$
x	0 1 1 1 1 1 1 1 1 1 1 1...	1
ϕ	0 1 0 1 0 1 0 1 0 1 0 1...	Φ
$\mathrm{J}(x)$	0 0 1 0 1 0 1 0 1 0 1 0...	$1+\Phi$

$\partial\mathcal{F}$	$\{0,1\}^\omega$	$\widehat{\mathbf{R}}$
x	1 0 0 0 0 0 0 0 0 0 0 0...	1
ϕ	0 1 0 1 0 1 0 1 0 1 0 1...	Φ
$\mathrm{J}(x)$	1 1 0 1 0 1 0 1 0 1 0 1...	$1/(1+\Phi)$

Conversely, one has $\mathrm{J}_{\mathbf{R}}(1+\Phi) = \mathrm{J}_{\mathbf{R}}(1/(1+\Phi)) = 1$, illustrating the two-to-oneness of $\mathrm{J}_{\mathbf{R}}$ on the set of noble numbers.

The noble numbers are the $\mathrm{PSL}_2(\mathbf{Z})$-translates of the golden section Φ. They correspond to the eventually zig-zag paths in $\partial\mathcal{F}$, and represented by strings terminating with $(01)^\omega$. From the $\veebar$-description, it is clear that $\mathrm{J}_{\partial\mathcal{F}}$ sends those strings to rational (i.e. eventually constant) strings and vice versa.

Since the equivalence $\sim_*$ is not respected by $\mathrm{J}_{\partial\mathcal{F}}$, it does not induce a homeomorphism of $\widehat{\mathbf{R}}$, not even a well-defined map. Nevertheless, if we ignore the pairs of rational ends, then the remaining equivalence classes are singletons and $\mathrm{J}_{\partial\mathcal{F}}$ restricts to a well-defined map

$$\mathrm{J}_{\mathbf{R}} : \widehat{\mathbf{R}} \setminus \widehat{\mathbf{Q}} \to \widehat{\mathbf{R}}$$

If we also ignore the set of "golden paths", i.e. the set $\mathrm{PGL}_2(\mathbf{Z})\widehat{\mathbf{Q}} = \text{ج}^{-1}(\mathbf{Q})$, then we obtain an involutive bijection

$$\text{ج}_{\mathbf{R}} : \widehat{\mathbf{R}} \setminus (\widehat{\mathbf{Q}} \cup \mathrm{PGL}_2(\mathbf{Z})\widehat{\mathbf{Q}}) \to \widehat{\mathbf{R}} \setminus (\widehat{\mathbf{Q}} \cup \mathrm{PGL}_2(\mathbf{Z})\widehat{\mathbf{Q}})$$

The functional equations. Since twists and shuffles do not change the distance to the base, we have our first functional equation:

Lemma 9 *The $|\mathcal{F}|$-automorphisms $\text{ج}_{\mathcal{F}}$ and $U = \sigma_{V_\bullet}$ commute, i.e. $\text{ج}_{\mathcal{F}}U = U\text{ج}_{\mathcal{F}}$. Hence, $\text{ج}_{\partial\mathcal{F}}U = U\text{ج}_{\partial\mathcal{F}}$ and whenever $\text{ج}_{\mathbf{R}}$ is defined, one has*

$$\text{ج}_{\mathbf{R}}U = U\text{ج}_{\mathbf{R}} \iff \text{ج}_{\mathbf{R}}\big(\frac{1}{x}\big) = \frac{1}{\text{ج}_{\mathbf{R}}(x)} \tag{16}$$

In fact, $\text{ج}_{\mathcal{F}}U$ is the automorphism of $\mathcal{F}$ which shuffles every other vertex, starting with the vertex v^*. In other words, it shuffles those vertices which are not shuffled by $\text{ج}_{\mathcal{F}}$. We denote this automorphism by $\text{ج}^*_{\mathcal{F}}$. Note that, in terms of the strings, the operation U is nothing but the term-wise negation:

$$Ua = \neg a,$$

and the lemma merely states the fact that $\neg(\phi\underline{\vee}a) = \phi\underline{\vee}\neg a$. The boundary homeomorphism induced by the automorphism $\text{ج}^*_{\mathcal{F}}$ is thus the map $\text{ج}^*_{\partial\mathcal{F}}$ which xors with the string $\phi^* := (10)^\omega$.

Now, consider the operation S. If $x = a \in \{0,1\}^\omega$, then one has $\text{ج}_{\partial\mathcal{F}}(Sx) =$

$$= S(a\underline{\vee}\phi^*) = S(a\underline{\vee}\neg\phi) = S(\neg(a\underline{\vee}\phi)) = S(U(a\underline{\vee}\phi)) = SU\text{ج}_{\partial\mathcal{F}}(x) = V\text{ج}_{\partial\mathcal{F}}(x)$$

The same equality holds if $x = Sa$, and we get our second functional equation:

Lemma 10 *The $\partial_I\mathcal{F}$-homeomorphisms $\text{ج}_{\partial\mathcal{F}}$ and S satisfy $\text{ج}_{\partial\mathcal{F}}S = V\text{ج}_{\partial\mathcal{F}}$. Hence, whenever $\text{ج}_{\mathbf{R}}$ is defined, one has*

$$\text{ج}_{\mathbf{R}}S = V\text{ج}_{\mathbf{R}} \iff \text{ج}_{\mathbf{R}}\big(-\frac{1}{x}\big) = -\text{ج}_{\mathbf{R}}(x) \tag{17}$$

Now we consider the operator L. Suppose that $x = Sa$. Then $Lx = LSa = 0a$. Hence, noting that $\phi = (01)^\omega = 0(10)^\omega = 0\phi^*$ we have

$$\text{ج}_{\partial\mathcal{F}}(Lx) = 0a\underline{\vee}\phi = 0a\underline{\vee}0\phi^* = 0(a\underline{\vee}\phi^*) = LS(a\underline{\vee}\phi^*) = L\text{ج}_{\partial\mathcal{F}}(x).$$

Another possibility is that $x = 0a$. In other words, x starts with an L. Then Lx starts with an L^2, i.e. $Lx = 1a$. Hence,

$$\text{ج}_{\partial\mathcal{F}}(Lx) = 1a\underline{\vee}\phi = 1a\underline{\vee}0\phi^* = 1(a\underline{\vee}\phi^*),$$
$$\text{ج}_{\partial\mathcal{F}}(x) = 0a\underline{\vee}\phi = 0a\underline{\vee}0\phi^* = 0(a\underline{\vee}\phi^*) \implies L\text{ج}_{\partial\mathcal{F}}(x) = 1(a\underline{\vee}\phi^*).$$

Finally, if $x = 1a$, then Lx starts with an S, i.e. $Lx = Sa$. Hence,

$$\mathfrak{J}_{\partial\mathcal{F}}(x) = 1a\underline{\vee}\phi = 1a\underline{\vee}0\phi^* = 1(a\underline{\vee}\phi^*) \implies \begin{aligned} \mathfrak{J}_{\partial\mathcal{F}}(Lx) &= S(a\underline{\vee}\phi^*), \\ L\mathfrak{J}_{\partial\mathcal{F}}(x) &= S(a\underline{\vee}\phi^*). \end{aligned}$$

Whence the third functional equation:

Lemma 11 *The $\partial_I\mathcal{F}$-homeomorphisms $\mathfrak{J}_{\partial\mathcal{F}}$ and L satisfy $\mathfrak{J}_{\partial\mathcal{F}}L = L\mathfrak{J}_{\partial\mathcal{F}}$. Hence, whenever $\mathfrak{J}_{\mathbf{R}}$ is defined, one has*

$$\mathfrak{J}_{\mathbf{R}}L = L\mathfrak{J}_{\mathbf{R}} \iff \mathfrak{J}_{\mathbf{R}}\big(1 - \frac{1}{x}\big) = 1 - \frac{1}{\mathfrak{J}_{\mathbf{R}}(x)} \tag{18}$$

(To be brief, the lemma holds true since L rotates paths around the vertex $v^* = \{I, L, L^2\}$ and so it does not change the distance of an edge to that vertex.)

Since U, S and L generate the group $\mathrm{PGL}_2(\mathbf{Z})$, these three functional equations forms a complete set of functional equations, from which the rest can be deduced. For example,

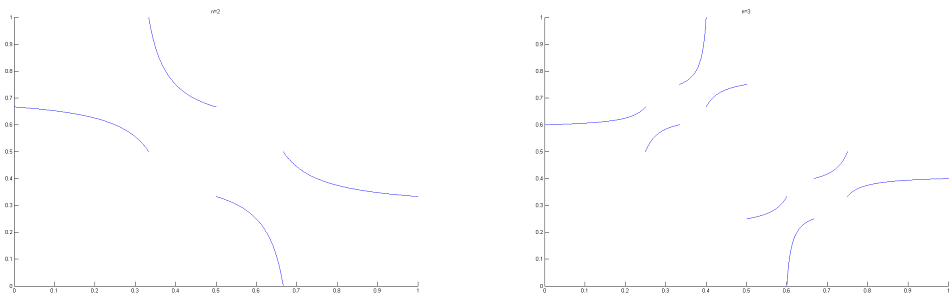
$$T = LS \implies \mathfrak{J}_{\partial\mathcal{F}}T = LV\mathfrak{J}_{\partial\mathcal{F}} \iff \mathfrak{J}_{\mathbf{R}}(1+x) = 1 + \frac{1}{\mathfrak{J}_{\mathbf{R}}(x)}$$

These functional equations also shows that $\mathfrak{J}_{\mathbf{R}}$ acts as the desired outer automorphism of $\mathrm{PGL}_2(\mathbf{Z})$.

Jimm as a limit of piecewise-$\mathrm{PGL}_2(\mathbf{Z})$ functions. In $\mathcal{F}$, denote by $V_\bullet(n)$ the set of vertices of distance $\leq n$ to the base (excluding v^*), and let $V'_\bullet(n)$ be the set of vertices of odd distance $\leq n$ to the base. Then one has

$$\mathfrak{J}_{\mathcal{F}} = \lim_{n\to\infty} \theta_{V_\bullet(n)} = \lim_{n\to\infty} \sigma_{V'_\bullet(n)}.$$

The maps $\theta_{V_\bullet(n)}$ and $\sigma_{V'_\bullet(n)}$ induce a sort of finitary projective interval exchange maps on $\widehat{\mathbf{R}}$ and thus $\mathfrak{J}_{\mathbf{R}}$ can be written as a limit of such functions. Below we draw the twists $\theta_{V_\bullet(n)}$ for $n = 1 \ldots 9$.



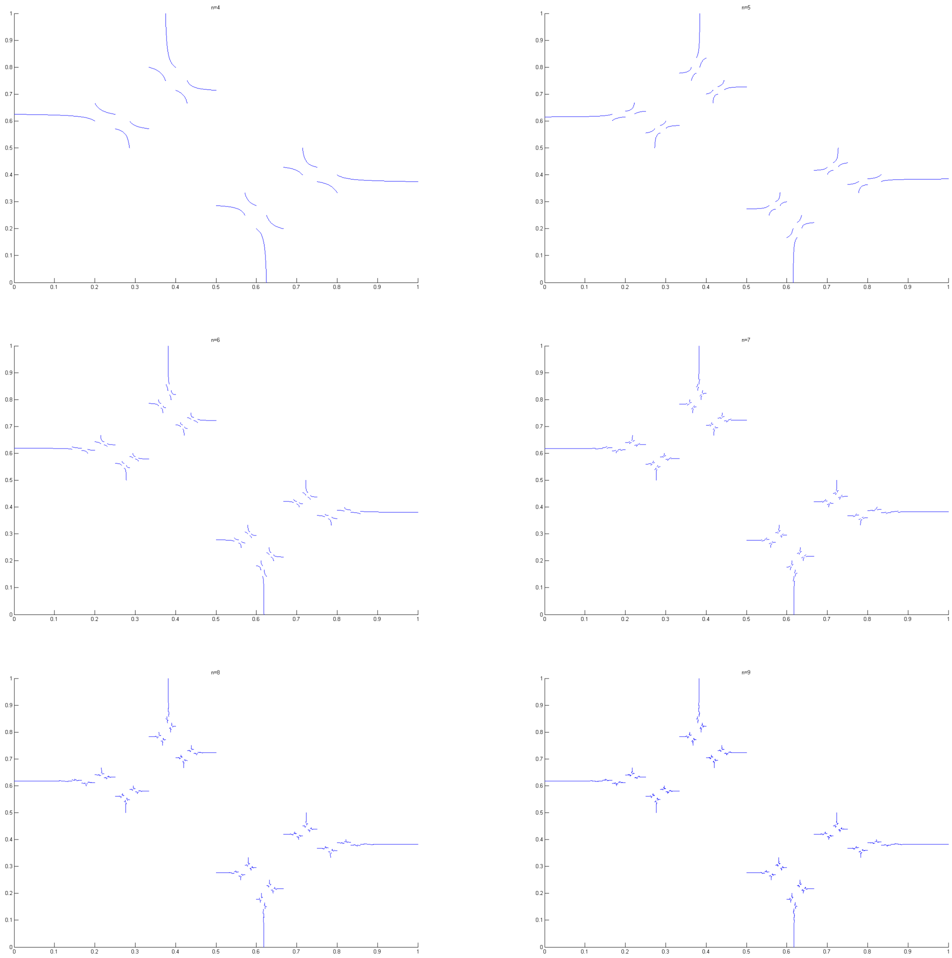


Figure. The plot of $\theta_{\mathcal{F}_n^*}$ on the interval $[0,1]$, for $n = 2, 3, \ldots 9$.

It is of interest to further study the effects of automorphisms of $|\mathcal{F}|$ on the boundary circle and see how they conjugate the dynamical system of the Gauss map. For example, any subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbf{Z})$ (or conjugacy class of a subgroup) corresponds to a set of vertices defining two automorphisms θ_Γ and σ_Γ. Any element γ of the boundary $\partial\mathcal{F}$ also defines a shuffle-automorphism σ_γ and a twist-automorphism θ_γ, constructed by using the set of vertices lying on the path connecting the base edge to γ.

Modular graphs. Since the modular group acts on the Farey tree, so does any subgroup Γ of the modular group, and the resulting quotient graph $\mathcal{F}/\Gamma$ of the Farey tree $\mathcal{F}$ is called a *modular graph*, see [47]. It is a bipartite graph with a ribbon structure. Conjugate subgroups give rise to isomorphic modular graphs. These are in fact special kind of dessins, the ambient punctured surface of a modular graph carries a canonical arithmetic structure.

The modular group is not invariant under $\mathfrak{J}$, however, the subgroup $\langle R, SRS\rangle \simeq \mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ is invariant and its subgroups are sent to each other under $\mathfrak{J}$. Hence $\mathfrak{J}$ acts on the corresponding modular graphs by $\mathcal{F}/\Gamma \to \mathcal{F}/\mathfrak{J}\Gamma$. This action shuffles (i.e. reverses the cyclic ordering of) every other trivalent vertex of the graph. The genus of the ambient surface is not invariant under this action. On the other hand, its fundamental group is invariant. Jones and Thornton [23] described this action in terms of the dual graphs (i.e. triangulations).The use of triangulations requires to cut the ambient surface into pieces, shuffle, and then glue the pieces back; which is somewhat hard to imagine for us.

On the other hand, note that metrization of the graphs $\mathcal{F}/\Gamma$ parametrize the decorated Teichmüller space of Penner [42] and we see that $\mathfrak{J}$ induces a certain duality on punctured Riemann surfaces. This duality do not respect the genus, but it does respect the rank of the fundamental group.

Jimm on $\mathbf{Q}\setminus\{0,1\}$. In virtue of Lemma 3, there is a 1-1 correspondence between the set of trivalent vertices and the set $\mathbf{Q}\setminus\{0,1\}$. Since any element of $Aut_I(\mathcal{F})$ defines a bijection of $V_\bullet(\mathcal{F})$, we see that every automorphism of $\mathcal{F}$ that fixes I, defines a unique bijection of $\mathbf{Q}\setminus\{0,1\}$. In particular, this is the case with $\mathfrak{J}_\mathcal{F}$. We denote the resulting bijection with $\mathfrak{J}_\mathbf{Q}$. One has $\mathfrak{J}_\mathbf{Q}(1) = 1$, and for $x > 0$, its values can be computed by using the functional equations $\mathfrak{J}_\mathbf{Q}(1+x) = 1+1/\mathfrak{J}_\mathbf{Q}(x)$ and $\mathfrak{J}_\mathbf{Q}(1/x) = 1/\mathfrak{J}_\mathbf{Q}(x)$. It tends to $\mathfrak{J}_\mathbf{R}$ at irrational points. However, it must be emphasised that $\mathfrak{J}_\mathbf{Q}(x)$ is not the restriction of $\mathfrak{J}_\mathbf{R}$ to $\mathbf{Q}$; this latter function is by definition two-valued at rationals[6]. The involution $\mathfrak{J}_\mathbf{Q}$ conjugates the multiplication on $\widehat{\mathbf{Q}}\setminus\{0,\infty\}$ to an operation with 1 as its identity, and such that the inverse of q is $1/q$.

3.3 Fun with functional equations

It is possible to derive many equations from the functional equations for $\mathfrak{J}$, or express them in alternative forms. In this section we record some of these equations. We leave the task of verifying to the reader. (As usual we drop the subscript $\mathbf{R}$ from $\mathfrak{J}_\mathbf{R}$ to increase the readability):

[6] It might have been convenient to declare the values of $\mathfrak{J}_\mathbf{R}$ at rational arguments to be given by $\mathfrak{J}_\mathbf{Q}(x)$, so that $\mathfrak{J}_\mathbf{R}$ would be a well-defined function everywhere (save 0 and ∞). However, we have chosen to not to follow this idea, for the sake of uniformity in definitions.

To start with, note that ج satisfies the following functional equations

$$\text{ج}\left(1-\frac{1}{x}\right)=1-\frac{1}{\text{ج}(x)},\quad \text{ج}\left(\frac{x}{x+1}\right)+\text{ج}\left(\frac{1}{x+1}\right)=1.$$

There is the two-variable version of the functional equations

$$\boxed{\begin{gathered}
\text{ج}(x)=y \iff \text{ج}(y)=x\\[2ex]
xy=1 \iff \text{ج}(x)\text{ج}(y)=1\\[2ex]
x+y=0 \iff \text{ج}(x)\text{ج}(y)=-1\\[2ex]
x+y=1 \iff \text{ج}(x)+\text{ج}(y)=1\\[2ex]
\frac{1}{x}+\frac{1}{y}=1 \iff \frac{1}{\text{ج}(x)}+\frac{1}{\text{ج}(y)}=1
\end{gathered}}$$

Functional equations can be expressed in terms of the involution $\text{ج}^*(x):=1/\text{ج}(x)=U\text{ج}(x)$:

$$\boxed{\begin{gathered}
\text{ج}^*(\text{ج}^*(x))=x\\[2ex]
xy=1 \iff \text{ج}^*(x)\text{ج}^*(y)=1\\[2ex]
x+y=0 \iff \text{ج}^*(x)\text{ج}^*(y)=-1\\[2ex]
x+y=1 \iff \frac{1}{\text{ج}^*(x)}+\frac{1}{\text{ج}^*(y)}=1\\[2ex]
\frac{1}{x}+\frac{1}{y}=1 \iff \text{ج}^*(x)+\text{ج}^*(y)=1\\[2ex]
\text{ج}^*(x+1)=\frac{1}{1+\text{ج}^*(x)}
\end{gathered}}$$

Finally, the functional equations can be expressed in terms of the function $\Psi(x):=\log|\text{ج}(x)|$, where they look like cocycle relations, compare [53].

$$\boxed{\begin{array}{c} \Psi(1/x) + \Psi(x) = 0 \\[2em] \Psi(x) + \Psi(1+x) + \Psi(x/(1+x)) = 0 \\[2em] \exp(\Psi(1-x)) + \exp(\Psi(x)) = 1 \end{array}}$$

(the involutive relation is left to the reader).

One is tempted to seek the solutions of (some of the) functional equations which are analytic in some region, the most appealing one being the equation $\Psi(1+x) = 1 + 1/\Psi(x)$. Beware that by iterating this equation we get

$$\lim_{x\to+\infty} \Psi(x) = \Phi \text{ and } \lim_{x\to-\infty} \Psi(x) = -1/\Phi, \tag{19}$$

which implies that there are no rational solutions. Here $\phi = (1+\sqrt{5})/2$ as usual. We don't know if there exists an analytic solution in some strip around the real line. Another temptation is to extend $\mathfrak{J}$ to the upper half plane in some sense, but we have no idea how this must be done.

Since $\mathfrak{J}_{\mathbf{R}}$ satisfies this functional equation, its limit at $\pm\infty$ exists and these limits are given by (19). See also Lemma 12 (ii) below.

4 Analytical aspects of jimm

4.1 Fibonacci sequence and the Golden section

The fabulous Fibonacci sequence is defined by the recurrence

$$F_0 = 0, \quad F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \in \mathbf{Z},$$

one has then $F_{-n} = (-1)^{n+1} F_n$. Here is a table of some small Fibonacci numbers:

…	F_6	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	$F_7 \dots$
…	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	$13\dots$

One has

$$\widetilde{T}(x) = 1 + \frac{1}{x} \implies \widetilde{T}^n = \frac{F_{n+1}x + F_n}{F_n x + F_{n-1}} \quad (n \in \mathbf{Z})$$

and therefore

$$\det(\widetilde{T}(x)) = -1 \implies \det(\widetilde{T}^n) = (-1)^n \implies F_{n+1}F_{n-1} - F_n^2 = (-1)^n \quad (n \in \mathbf{Z})$$

The transformation $\widetilde{T}$ has two fixed points, solutions of the equation

$$\widetilde{T}(x) = x \iff 1 + \frac{1}{x} = x \iff x^2 - x - 1 = 0 \iff x = \frac{1 \pm \sqrt{5}}{2} \tag{20}$$

As we already mentioned, the positive root of this equation is called the golden section which we shall denote by Φ. It is the limit $\Phi = \lim_{n\to\infty} F_{n+1}/F_n$. The negative root of Equation (20) equals $\Phi_* := -1/\Phi$. The numbers F_n grow exponentially as $F_n = (\Phi^n - \Phi_*^n)/\sqrt{5}$ by Binet's formula. This fact will be used in the following form

$$F_n > 1\sqrt{5}\Phi^n \implies F_n^{-1} < \sqrt{5}\Phi^{-n} \tag{21}$$

The following lemma is an easy consequence of the preceding discussions.

Lemma 12 *Let x be an irrational number or an indeterminate. Then*
(i)

$$\mathfrak{T}(1+x) = 1 + \frac{1}{\mathfrak{T}(x)} \iff \mathfrak{T}(Tx) = \widetilde{T}\mathfrak{T}(x).$$

(ii)

$$\mathfrak{T}(n+x) = \widetilde{T}^n\mathfrak{T}(x) = \frac{F_{n+1}\mathfrak{T}(x) + F_n}{F_n\mathfrak{T}(x) + F_{n-1}} \quad (n \in \mathbf{Z}).$$

(iii)

$$\mathfrak{T}([1,x]) = \mathfrak{T}(1 + \frac{1}{x}) = 1 + \mathfrak{T}(x) \implies \mathfrak{T}([1_n, x]) = n + x \quad (n \in \mathbf{Z}).$$

(iv)

$$\mathfrak{T}([n,x]) = \mathfrak{T}(n + 1/x) = \frac{F_{n+1} + F_n\mathfrak{T}(x)}{F_n + F_{n-1}\mathfrak{T}(x)} \quad (n \in \mathbf{Z}).$$

Lemma 13 *One has $0 \le \mathfrak{T}(x) \le 1$ if $0 \le x \le 1$, with $\mathfrak{T}(1/\Phi) = 0$, $\mathfrak{T}(1/(1+\Phi)) = 1$, $\mathfrak{T}(1/(n+\Phi)) = F_n/F_{n+1}$. Hence $\mathfrak{T}$ restricts to an involution of the unit interval. More generally, $\mathfrak{T}$ maps Farey intervals to Farey intervals.*

The lemma below is also a routine observation.

Lemma 14 *If x is given by the "minus-continued fraction"*

$$x = n_0 - \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{\dots}}} = T^{n_0} \circ UV \circ T^{n_1} \cdot UV \dots,$$

then $\mathfrak{T}(x)$ is given by the continued fraction

$$\mathfrak{T}(x) = \widetilde{T}^{n_0} \circ V \circ \widetilde{T}^{n_1} \circ V \circ \widetilde{T}^{n_2} \circ \dots .$$

Finally, one has the following simple observation.

Lemma 15 *The following is a* $\mathrm{PGL}_2(\mathbf{Z})$*-invariant of four-tuples of real numbers*

$$\big[x : y : z : t\big]_{\text{Ϛ}} := \big[\text{Ϛ}(x) : \text{Ϛ}(y) : \text{Ϛ}(z) : \text{Ϛ}(t)\big],$$

where $\big[x : y : z : t\big]$ *denotes the cross ratio.*

4.2 Continuity and jumps

Since the involution $\text{Ϛ}_{\partial\mathcal{F}} : \partial\mathcal{F} \to \partial\mathcal{F}$ is a homemorphism and since the subspaces

$$\mathbf{R} \setminus \mathbf{Q} \simeq \partial\mathcal{F} \setminus \{\text{rational paths}\}$$

are also homeomorphic, the function $\text{Ϛ}_{\mathbf{R}}$ is continuous at irrational points. Let us record this fact.

Theorem 16 *The function* $\text{Ϛ}_{\mathbf{R}}$ *on* $\mathbf{R} \setminus \mathbf{Q}$ *is continuous.*

Moreover, recall that there is a canonical ordering on $\partial\mathcal{F}$ inducing on $\widehat{\mathbf{R}}$ its canonical ordering compatible with its topology. So the notion of lower and upper limits exists on $\partial\mathcal{F}$ and coincides with lower and upper limits on $\widehat{\mathbf{R}}$. This shows that the two values that $\text{Ϛ}_{\mathbf{R}}$ assumes on rational arguments are nothing but the limits

$$\text{Ϛ}(q)^- := \lim_{x \to q^-} \text{Ϛ}(x) \text{ and } \text{Ϛ}(q)^+ := \lim_{x \to q^+} \text{Ϛ}(x)$$

By choosing one of these values coherently, one can make Ϛ an everywhere upper (or lower) continuous function. Note however that it will (partially) cease to satisfy the functional equations at rational arguments. One has, for irrational r and rational q,

$$\lim_{q \to r} \text{Ϛ}_{\mathbf{R}}(q) = \text{Ϛ}_{\mathbf{R}}(r),$$

no matter how we choose the values of $\text{Ϛ}_{\mathbf{R}}(q)$.

The jump function

$$\delta_{\text{Ϛ}}(q) = \text{Ϛ}(q)^+ - \text{Ϛ}(q)^-$$

can be considered as a (signed) measure of complexity of a rational number q. For irrational numbers it is 0. It splits the set $\mathbf{Q}$ into two pieces, according to the sign of $\delta_{\text{Ϛ}}(q)$.

The following functional equations are easily deduced from those for Ϛ:

$$
\boxed{\begin{gathered}
\delta_{\text{ج}}(-1/x) = -\delta_{\text{ج}}(x) \iff \delta_{\text{ج}}(-x) = -\delta_{\text{ج}}(1/x) \\[2ex]
\delta_{\text{ج}}(1-x) = \delta_{\text{ج}}(x) \iff \delta_{\text{ج}}(1+x) = \delta_{\text{ج}}(-x) \\[2ex]
\delta_{\text{ج}}(-x) = -\frac{\delta_{\text{ج}}(x)}{\text{ج}^{+}(x)\text{ج}^{-}(x)}
\end{gathered}}
$$

Hence $|\delta_{\text{ج}}|$ is invariant under the infinite dihedral group $\langle -1/x, 1-x\rangle$. In order to compute some values of $\delta_{\text{ج}}(q)$ more explicitly, one has, for k odd,

$$
\begin{gathered}
q = [n_0, n_1, \dots n_k, \infty] = [n_0, n_1, \dots n_k - 1, 1, \infty] \implies \\
\text{ج}(q)^{+} = \text{ج}([n_0, n_1, \dots n_k, \infty]), \quad \text{ج}(q)^{-} = \text{ج}([n_0, n_1, \dots n_k - 1, 1, \infty]) \\
\implies \delta_{\text{ج}}(q) = \text{ج}([n_0, n_1, \dots n_k, \infty]) - \text{ج}([n_0, n_1, \dots n_k - 1, 1, \infty])
\end{gathered}
$$

(and for k even, $\delta_{\text{ج}}(q)$ changes sign).

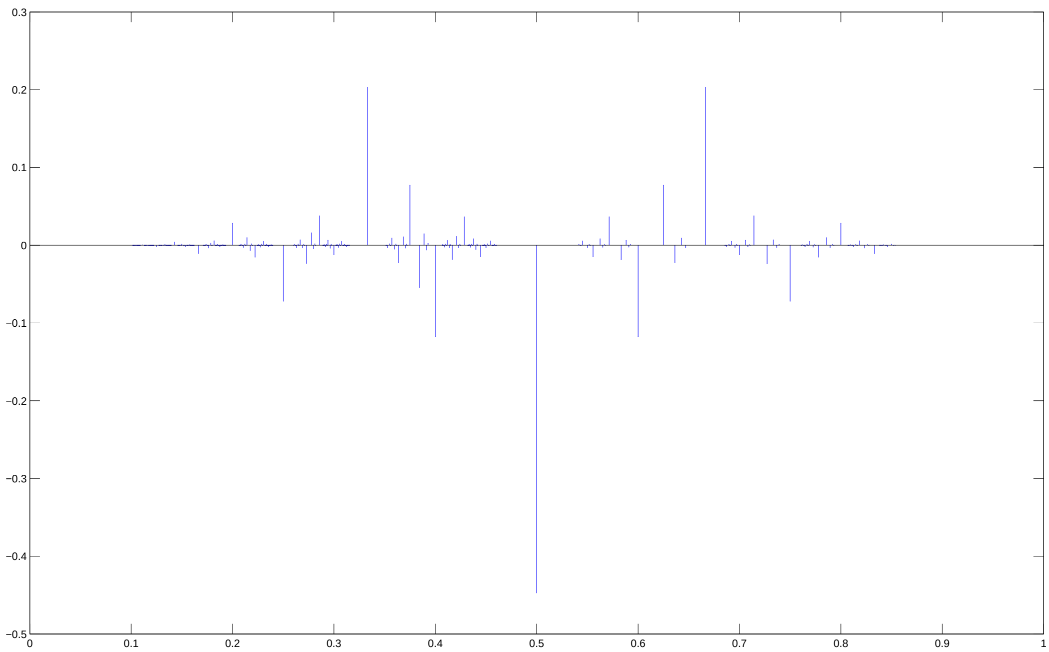


Figure. The plot of $\delta_{\text{ج}}$, centered around the point $x = 1/2$.

Suppose $q = n$ is an integer. Then

$$
\text{ج}(n)^{+} = \frac{F_n\Phi + F_{n+1}}{F_{n-1}\Phi + F_n}
$$

$$\text{?}(n)^{-} = \frac{F_{n-1}\Phi + F_n}{F_{n-2}\Phi + F_{n-1}}$$

More conceptually, one has

$$\text{?}[n, x] = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \text{?}[n-k, x] \text{ for } 0 \leq k \leq n.$$

Hence,

$$\text{?}[n, \infty] = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \text{?}[1, \infty], \quad \text{?}[n-1, 1, \infty] = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \text{?}[0, 1, \infty].$$

Since

$$\text{?}[1, \infty] = 1 + 1/\text{?}[0, \infty] = 1 + \text{?}[\infty] = 1 + \Phi = \Phi^2$$

and

$$\text{?}[0, 1, \infty] = 1/\text{?}[1, \infty] = 1/(1+\Phi) = \Phi^{-2}$$

we

$$\begin{aligned} \delta_{\text{?}}(n) &= \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \Phi^2 - \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \Phi^{-2} \\ &= (-1)^{n+1} \frac{\Phi^2 - \Phi^{-2}}{(F_{n-1}\Phi^2 + F_{n-2})(F_{n-1}\Phi^{-2} + F_{n-2})} \end{aligned}$$

and we finally have

$$\delta_{\text{?}}(n) = (-1)^{n+1} \frac{\sqrt{5}}{(F_{n-1}^2 + 3F_{n-1}F_{n-2} + F_{n-2}^2)} = (-1)^{n+1} \frac{\sqrt{5}}{F_n^2 + F_{n-1}F_{n-2}}$$

We see that this rapidly tends to zero as $n \to \infty$.

Now, it should be true that ? is of bounded variation on any finite or infinite closed interval not containing Φ nor $-\Phi^{-1}$, and that the signed measure $d\text{?}$ is just the function $\delta_{\text{?}}$, but we defer the proofs out of weariness. On the other hand, it is proved in Theorem 25 the derivative of ? exists vanishes almost everywhere.

Concerning the integrals of ?, below is an easy consequence of the functional equation $\text{?}K = K\text{?}$.

Lemma 17 *One has*

$$\int_0^1 \text{?}(x)dx = \frac{1}{2}, \quad \int_0^t \text{?}(x)dx = t - 1/2 + \int_0^{1-t} \text{?}(x)dx$$

By the change of variable $y = 1/x$ this gives

$$\int_1^\infty \frac{dy}{y^2 \text{?}(y)} = \frac{1}{2}$$

We failed to compute other integrals (moments, transforms, etc). On the other hand, as we remarked in the introduction we have developed some numerical algorithms and implemented them on a computer for the interested reader.

4.3 The graph of jimm

Since $\mathfrak{J}_{\mathbf{R}}$ has a non-removable discontinuity at every rational point, it is not possible to draw its graph in the usual way. Below is a computer-produced box-graph of $\mathfrak{J}(x)$ for $x \geq 0$, the graph lies inside the smaller (and darker) boxes.

This graph is found as follows. If $x \in (1,2)$, then $x = [1, n_1, r]$ with $r \in (1, \infty)$, and there are two cases

$$x \in (1,\infty) = \bigcup_{n_0=1}^{\infty} (n_0, \infty) \iff x = [n_0, r] \text{ with } n_0 \geq 1, \text{ and } r \in (1,\infty)$$

$$\begin{aligned} (n_0 \geq 1), \quad x \in (n_0, n_0+1) = & \bigcup_{n_1=1}^{\infty} \left[n_0 + \frac{1}{n_1+1}, n_0 + \frac{1}{n_1} \right] \\ & x = [n_0, n_1, r] \text{ with } n_1 \geq 1, \quad 1 < r < \infty \\ \implies & \mathfrak{J}(x) = [\underbrace{1,1,\ldots,1}_{n_0-1\, times}, m_1, \underbrace{1,1,\ldots,1}_{n_1-2\, times}, \ldots, 1, 2, \ldots], \end{aligned}$$

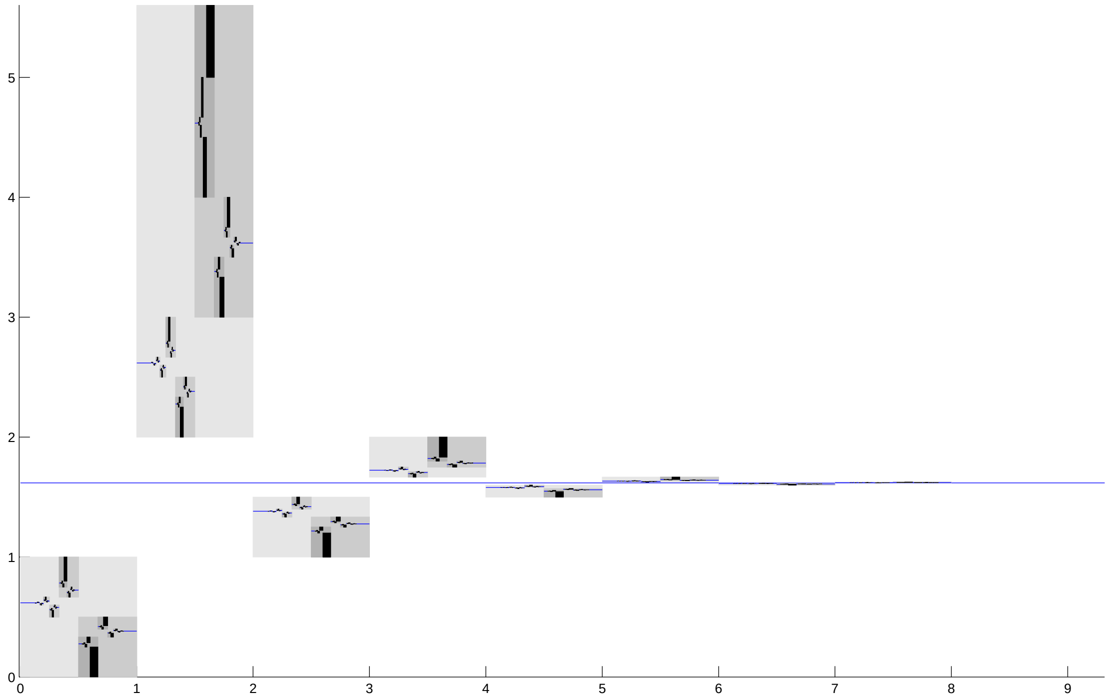
where m_1=2 if $n_1 > 1$ and $m_1 \geq 3$ if $n_1 = 1$. So in any case, $m_1 \geq 2$, so that

$$(n_0 \geq 1), \quad x \in (n_0, n_0+1) \implies \mathfrak{J}(x) \in [\underbrace{1,1,\ldots,1}_{n_0-1\, times}, [2,\infty]]$$

In other words,

$$(n_0 \geq 2), \implies \quad \mathfrak{J}((n_0, n_0+1)) = \left[\frac{F_{n_0+3}}{F_{n_0+2}}, \frac{F_{n_0+1}}{F_{n_0}} \right]$$

(intervals being inverted depending on the parity of n_0). Now suppose that $x \in [2,3)$. Continuing in this manner gives the following graph:



The graph on the negative sector can be found by using the functional equation $\Game(-x) = -1/\Game(x)$.

5 Action on the quadratic irrationalities

Since $\Game_{\mathcal{F}}$ sends eventually periodic paths to eventually periodic paths, $\Game_{\mathbf{R}}$ preserves the real-multiplication set. This is the content of our next result:

Theorem 18 $\Game$ *defines an involution of the set of real quadratic irrationals, (including rationals) and it respects the* $\mathrm{PGL}_2(\mathbf{Z})$*-orbits.*

Beyond this theorem, we failed to detect any further arithmetic structure or formula relating the quadratic number to its $\Game$-transform. All we can do is to give some sporadic examples, which amounts to exhibit some special real quadratic numbers with known continued fraction expansions. We shall do this below. Before that, however, note that the equation below is solvable for every $n \in \mathbf{Z}$:

$$\Game x = n + x \implies x = \Game\Game x = \widetilde{T}^n \Game x \implies x = \widetilde{T}^n(x+n)$$

$$\frac{F_{n+1}(x+n) + F_n}{F_n(x+n) + F_{n-1}} = x = \frac{F_{n+1}x + F_n + nF_{n+1}}{F_n x^n + F_{n-1} + nF_n}$$

$$\Longrightarrow\ F_n x^2 + (F_{n-1} + nF_n - F_{n+1})x - (F_n + nF_{n+1})$$

More generally, the equation $\text{ج}x = Mx$ is solvable for $M \in \mathrm{PGL}_2(\mathbf{Z})$. Indeed one has

$$\text{ج}x = Mx \implies x = \text{ج}\text{ج}x = (\text{ج}M)(\text{ج}x) = (\text{ج}M)Mx,$$

and the solutions are the fixed points of $(\text{ج}M)M$. These points are precisely the words represented by the infinite path

$$x = (\text{ج}M)M(\text{ج}M)M(\text{ج}M)M(\text{ج}M)M\dots \tag{22}$$

where we assume that both M and $\text{ج}M$ are expressed as words in U and T. Note that these words are precisely the points whose conjugacy classes remain stable under ج.

$$\text{ج}x = M(\text{ج}M)M(\text{ج}M)M(\text{ج}M)M(\text{ج}M)M\dots$$

One may also view these numbers x as sort of eigenvectors of the ج operator: $\text{ج}(x) = M\text{ج}(x)$.

Hence the $\mathrm{PGL}_2(\mathbf{Z})$-orbits of the points in (22) remain stable under ج. Since the set of these orbits is the moduli space of real lattices, we deduce the result:

Proposition 19 *The fixed points of the involution ج on the moduli space of real lattices are precisely the points (22) .*

For example, if $\text{ج}M = \widetilde{T}^{k+1}$ then $M = T^{k+1}$, and the point (22) is nothing but the point $[\overline{1_k, k+2}]$ mentioned in the introduction.

Example. In [44] it is shown that

$$x = [0; \overline{1_{n-1}, a}] = \frac{a}{2}\left(\sqrt{1 + 4\frac{aF_{n-1} + F_{n-2}}{a^2F_n}} - 1\right)$$

Hence $\text{ج}(x) = [0; n, \overline{a-2, n+1}]$, *and* so $\text{ج}(x) = (n + \frac{1}{y})^{-1}$, where

$$y = \frac{(a-2)(n+1) + \sqrt{(a-2)^2(n+1)^2 + 4(a-2)(n+1)}}{2(n+1)}.$$

Example. (From Einsiedler & Ward [15], Pg.90, ex. 3.1.1) This result of McMullen from [34] illustrates how $\mathfrak{C}$ behaves on one real quadratic number field. $\mathbf{Q}(\sqrt{5})$ contains infinitely many elements with a uniform bound on their partial quotients, since $[\overline{1_{k+1}, 4, 5, 1_k, 3}] \in \mathbf{Q}(\sqrt{5}), \quad \forall k = 1, 2, \ldots$. Routine calculations shows that the transforms $\mathfrak{C}([\overline{1_{k+1}, 4, 5, 1_k, 3}]) = [k+2, \overline{2, 2, 3, k+2, 1, k+3}]$ lies in different quadratic number fields for different k's. Hence, not only $\mathfrak{C}$ does not preserve the property of "belonging to a certain quadratic number field", it sends elements from one quadratic number field to different quadratic number fields. The following result provides even more examples of this nature:

Theorem 20 *(McMullen [34]) For any $s > 0$, the periodic continued fractions*

$$x_m = [\overline{(1, s)^m, 1, s+1, s-1, (1, s)^m, 1, s+1, s+3)}]$$

lie in $\mathbf{Q}(\sqrt{s^2 + 4s})$ for any $m \geq 0$.

The fact recently shown by McMullen is that $\mathbf{Q}(\sqrt{d})$ contains infinitely elements with partial quotients bounded by some number M_d. He also raises the question: can one take $M_d = 2$?

Example 4 (Powers of the golden ratio)

Theorem 21 *(Fishman and Miller [17]) One has the following*

$$\Phi^k = \begin{cases} [\overline{F_{k+1} + F_{k-1}}], & \text{if } k \text{ is even,} \\ [F_{k+1} + F_{k-1} - 1, \overline{1, F_{k+1} + F_{k-1} - 2}] & \text{if } k \text{ is odd.} \end{cases}$$

Corollary 22 *The continued fraction of the kth power of the golden section is*

$$\mathfrak{C}(\Phi^k) = \begin{cases} [1_{F_{k+1}+F_{k-1}-1}, \overline{2, 1_{F_{k+1}+F_{k-1}-2}}], & \text{if } k \text{ is even,} \\ [1_{F_{k+1}+F_{k-1}-2}, \overline{3, 1_{F_{k+1}+F_{k-1}-4}}] & \text{if } k \text{ is odd.} \end{cases}$$

The following generalization of this is also known:

Theorem 23 *(Fishman and Miller [17]) Consider the recurrence relation*

$$g_{n+1} = mg_n + lg_{n-1}, \text{ with } g_0 = 0, g_1 = 1, \quad , m \in \mathbf{Z}_{>0}, \quad l = \pm 1.$$

Let

$$\Phi_{m,l} := \lim_{n \to \infty} \frac{g_n}{g_n - 1} = \frac{m \pm \sqrt{m^2 + 4l}}{2}.$$

Then for any positive integer k, if $l = 1$ one has

$$\Phi_{m,1}^k = \begin{cases} [\overline{g_{k+1} + g_{k-1}}], & \text{if } k \text{ is odd,} \\ [g_{k+1} + g_{k-1} - 1, \overline{1, g_{k+1} + g_{k-1} - 2}] & \text{if } k \text{ is odd,} \end{cases}$$

while if $l = 1$ one has

$$\Phi_{m,-1}^k = [g_{k+1} - g_{k-1} - 1, \overline{1, g_{k+1} - g_{k-1} - 2}] \quad \text{if } k \text{ is odd,}$$

We leave it to the reader to determine the $\mathfrak{T}$-transform of $\Phi_{m,l}^k$.

Example. Suppose $\alpha = p + \sqrt{q}$ be a quadratic irrational with $p, q \in \mathbf{Q}$. Then $\alpha^* = 1/\alpha$ provided

$$|\alpha|^2 = p^2 - q = 1 \implies q = p^2 - 1.$$

Suppose α is of this form, i.e. $\alpha = p + \sqrt{p^2 - 1}$ and suppose $\mathfrak{T}(\alpha) = x + \sqrt{y}$. Then since

$$x - \sqrt{y} = \mathfrak{T}(\alpha)^* = \mathfrak{T}(\alpha^*) = \mathfrak{T}(1/\alpha) = 1/\mathfrak{T}(\alpha) = 1/(x + \sqrt{y}) \implies x^2 - y = 1,$$

and we conclude that $\mathfrak{T}(\alpha)$ is again of the form $x + \sqrt{x^2 - 1}$.

On the other hand, suppose $\mathfrak{T}(\sqrt{q}) = x + \sqrt{y}$. Then

$$\mathfrak{T}(\sqrt{q})^* = \mathfrak{T}(\sqrt{q}^*) = \mathfrak{T}(-\sqrt{q}) = -1/\mathfrak{T}(\sqrt{q}) \implies |\mathfrak{T}(\sqrt{q}^*)| = x^2 - y = -1$$

Hence, $\mathfrak{T}(\sqrt{q})$ is of the form $x + \sqrt{1 + x^2}$. Conversely, if α is of the form $p + \sqrt{p^2 + 1}$, then $\mathfrak{T}(\alpha)$ is a quadratic surd.

6 Statistics

By the Gauss-Kuzmin theorem, for almost all real numbers x, the frequency of an integer $k > 0$ in the continued fraction expansion of x converges to

$$\frac{1}{\log 2} \log\left(1 + \frac{1}{k(k+2)}\right)$$

(see [21]). We say that x obeys the Gauss-Kuzmin statistics if this is the case.

If $X \in [0, 1]$ is a uniformly distributed random variable, then almost everywhere the arithmetic mean of its partial quotients tends to infinity, i.e. if $X = [0, n_1, n_2, \dots]$ then

$$\lim_{k \to \infty} \frac{n_1 + \cdots + n_k}{k} = \infty \qquad \text{(a.s.)} \tag{23}$$

(see [21]). In other words, the set of numbers in the unit interval such that the above limit is infinite, is of full Lebesgue measure. Denote this set by A.

Now since the first k partial fractions of X give rise to at most $n_1+\cdots+n_k-k$ partial fractions of $\text{ج}(X)$ and at least $n_1+\cdots+n_k-2k$ of these are 1's, one has

$$\frac{n_1+\cdots+n_k-k}{n_1+\cdots+n_k-2k} \to \frac{\frac{n_1+\cdots+n_k}{k}-1}{\frac{n_1+\cdots+n_k}{k}-2} \to 1$$

Lemma 24 *The density of* 1*'s in the continued fraction expansion of* $\text{ج}(X)$ *equals 1 a.e., and therefore the the continued fraction averages of* $\text{ج}(X)$ *tend to 1 a.e.*

We conclude that $\text{ج}(A)$ is a set of zero measure. This says that, even though the involution ج is a fundamental symmetry of $\mathbf{R}$, many properties of reals are strongly biased under it, unlike the other fundamental involutions U, V and K.

We leave open the questions such as: what is the distribution of the continued fraction entries of $\text{ج}(X)$, if we ignore the 1's?

6.1 The derivative of jimm.

A function discontinuous on a dense subset of $[0,1]$ can not be differentiable everywhere on the residual set; it can be differentiable at most on a meager set (i.e. a countable union of nowhere dense sets), see Fort's paper [18]. But meager does not mean negligible: There exists meager sets of full Lebesgue measure, and it was also exhibited in the location cited a function discontinuous at rationals and yet differentiable on a set of full measure. It turns out that ج is a function of this kind.

To see why, recall that average values of continued fraction elements tend to infinity almost everywhere. Let $a=[0,n_1,n_2,\ldots]$ be a number with this property:

$$\lim_{k\to\infty}\frac{n_1+\cdots+n_k}{k}=\infty. \tag{24}$$

Then for every constant M, there is some k with $n_1+\cdots+n_k>kM$. But then the ج- transform of the initial length-k segment of x is of length at least $kM-k$. Hence if y is any number whose continued fraction expansion coincide with that of x up to the place k, then the continued fraction $\text{ج}(y)$ coincide with that of $\text{ج}(x)$ at least up to the place $kM-k$. Since $kM-k$ is arbitrarily big compared to k, and since longer continued fractions give exponentially better approximations, we see that $\text{ج}(y)$ a.e. is much closer to $\text{ج}(x)$ then y is to x. Hence the idea of the following theorem.

Theorem 25 *The derivative of* $\mathfrak{C}(a)$ *exists almost everywhere and vanishes almost everywhere.*

To prove this, we need to show that for almost all a,

$$\lim_{x \to a} \frac{\mathfrak{C}(a) - \mathfrak{C}(x)}{a - x} = 0.$$

Assume that x is irrational or equivalently its continued fraction expansion is non-terminating.

Let $x \in [0, 1]$ with $0 < |x - a| < \delta$ for some δ. Then there is a number $k = k_\delta$, such that the continued fractions of a and x coincide up to the kth element. Hence $x = [0, n_1, n_2, \dots n_k, m_{k+1}, \dots]$ with $m_{k+1} \neq n_{k+1}$. Note that this latter condition also guarantees that $0 < |x - a|$. Now let

$$M_k(z) := [n_1, n_2, \dots, n_{k-1}, n_k + z] = \frac{\alpha_k z + \beta_k}{\gamma_k z + \theta_k}$$

and put $a_k := [0, n_{k+1}, n_{k+2}, \dots], \quad x_k := [0, m_{k+1}, m_{k+2}, \dots]$. Then one has $0 < a_k < 1$ (with strict inequality since a is irrational) and $0 \leq x_k < 1$ for every $k = 1, 2, \dots$. One has

$$a = M_k(a_k), \quad x = M_k(x_k) \text{ and } \det(M_k) = (-1)^k.$$

Lemma 26 *Let* $a := [0, n_0, n_1, \dots]$ *and suppose that the continued fractions of* a *and* x *coincide up to the place* k *(but not* $k + 1$*), where* $x \in [0, 1]$*. Put* $N_k := \sum_{i=1}^{k} n_i$*, and* $\mu_k := N/k$*. Then*

$$|a - x| > \frac{1}{24} (2\mu_{k+3})^{-2(k+3)}$$

Proof. One has

$$|a-x| = |M_k(a_k) - M_k(x_k)| = \left| \frac{\alpha_k a_k + \beta_k}{\gamma_k a_k + \theta_k} - \frac{\alpha_k x_k + \beta_k}{\gamma_k x_k + \theta_k} \right| = \frac{1}{(\gamma_k a_k + \theta_k)(\gamma_k x_k + \theta_k}$$)

Since

$$M_{i+1}(z) = M_i \left(\frac{1}{n_{i+1} + z} \right) = \frac{\beta_i z + (\alpha_i + n_{i+1} \beta_i)}{\theta_i z + (\gamma + n_{i+1} \theta_i)},$$

one has $\gamma_{i+1} = \theta_i$ and $\theta_{i+1} = \gamma_i + n_{i+1} \theta_i$. Hence $\theta_{i+1} > \gamma_{i+1} \implies \theta_{i+1} > \theta_i(1 + n_{i+1})$. This implies

$$\theta_i < (1 + n_1)(1 + n_2) \dots (1 + n_i),$$
$$\gamma_i < (1 + n_1)(1 + n_2) \dots (1 + n_{i-1}).$$

Since $0 \leq a_k, x_k < 1$, this implies

$$\gamma_k a_k + \theta_k < \gamma_k + \theta_k < 2(1+n_1)(1+n_2)\dots(1+n_k),$$
$$\gamma_k x_k + \theta_k < \gamma_k + \theta_k < 2(1+n_1)(1+n_2)\dots(1+n_k).$$

Hence, we get

$$|a - x| > \frac{|a_k - x_k|}{4(1+n_1)^2(1+n_2)^2\dots(1+n_k)^2}$$

To estimate $|a_k - x_k|$, consider

$$a_k - x_k = \frac{1}{n_{k+1} + a_{k+1}} - \frac{1}{m_{k+1} + x_{k+1}} = \frac{m_{k+1} - n_{k+1} + x_{k+1} - a_{k+1}}{(n_{k+1} + a_{k+1})(m_{k+1} + x_{k+1})}$$
$$> \frac{m_{k+1} - n_{k+1} + x_{k+1} - a_{k+1}}{(1 + n_{k+1})(1 + m_{k+1})}.$$

Now, if $m_{k+1} < n_{k+1}$ then set $m_{k+1} = n_{k+1} - t$ with $t \geq 1$. Then one has

$$|a_k - x_k| > \frac{|-t + x_{k+1} - a_{k+1}|}{(1 + n_{k+1})(1 + n_{k+1} - t)} > \frac{a_{k+1}}{(1 + n_{k+1})^2}$$

On the other hand, if $3n_{k+1} \geq m_{k+1} > n_{k+1}$ then

$$|a_k - x_k| > \frac{|1 + x_{k+1} - a_{k+1}|}{(1 + n_{k+1})(1 + 3n_{k+1})} > \frac{1 - a_{k+1}}{3(1 + n_{k+1})^2},$$

and if $m_{k+1} > 3n_{k+1}$ then

$$|a_k - x_k| = \frac{1 - \frac{n_{k+1}}{m_{k+1}} + \frac{x_{k+1}}{m_{k+1}} - \frac{a_{k+1}}{m_{k+1}}}{(1 + n_{k+1})(1 + \frac{1}{m_{k+1}})} > \frac{1}{6(1 + n_{k+1})}$$

So one has

$$|a_k - x_k| > \frac{a_{k+1}(1 - a_{k+1})}{6(1 + n_{k+1})^2},$$

which gives the estimation from below

$$|a - x| > \frac{a_{k+1}(1 - a_{k+1})}{24(1+n_1)^2(1+n_2)^2\dots(1+n_k)^2(1+n_{k+1})^2},$$

estimation obtained under the assumption that the continued fraction expansions of x and a coincide up until the kth term and differ for the $k+1$th term.

Now we have the crude estimate

$$\frac{1}{n_{k+2} + \cfrac{1}{n_{k+3} + 1}} > a_{k+1} > \frac{1}{1 + n_{k+2}} \implies a_{k+1}(1 - a_{k+1}) > \frac{1}{(1 + n_{k+2})^2} \frac{1}{(1 + n_{k+3})^2}.$$

which gives

$$|a-x| > \frac{1}{24(1+n_1)^2(1+n_2)^2 \dots (1+n_{k+2})^2(1+n_{k+3})^2},$$

Now put $N_k := \sum_{i=1}^k n_i$, and $\mu_k := N/k$. Then

$$(1+n_1)^2(1+n_2)^2 \dots (1+n_k)^2 \leq (1+\mu_k)^{2k} \leq (2\mu_k)^{2k}$$

The last inequality follows from the fact that $\mu_k \geq 1$ for all k, since $n_i \geq 1$ for all i. We finally obtain the estimate

$$|a-x| > \frac{1}{24}(2\mu_{k+3})^{-2(k+3)} = \frac{1}{24}\exp\{-2(k+3)\log 2\mu_{k+3}\}$$

□

On the other hand, if the c.f. expansions of a and x coincide up to the $k = k(x)$th place, then the c.f. expansions of $\mathfrak{C}(a)$ and $\mathfrak{C}(x_i)$ coincide up to the place N_k, and by (21) we have

$$|\mathfrak{C}(a) - \mathfrak{C}(x)| < F_{N_k}^{-2} < \sqrt{5}\phi^{-2N_k} = \sqrt{5}\exp\{-2k\mu_k \log\phi\}$$

(This estimate should be close to optimal (a.e.), since the density of 1's in the c.f. expansion of $\mathfrak{C}(a)$ is 1 (a.e.) by Lemma 24.) This gives

$$\left|\frac{\mathfrak{C}(a)-\mathfrak{C}(x)}{a-x}\right| < 24\sqrt{5}\exp k\{2(1+3/k)\log 2\mu_{k+3} - 2\mu_k\log\phi\} \implies$$

$$\left|\frac{\mathfrak{C}(a)-\mathfrak{C}(x)}{a-x}\right| < A\exp\Big\{2k\log\phi\big(B\log 2\mu_{k+3} - \mu_k\big)\Big\}$$

where A is some absolute constant and $B = (1+3/k)/\log\phi$ can be taken arbitrarily close to $1/\log\phi < 2.08$ by assuming k is big enough.

We see immediately that, if $a = [0, n, n, n, n, \dots]$ then μ_k is constant $= n$, and if n is taken big enough so that $2.08\log 2n - n < 0$, then the derivative exists and is zero. This is true for $n > 4$. We don't claim that our estimations are optimal in this respect, however.

On the other hand, since $\mu_k \to \infty$ almost surely, we see that $B\log 2\mu_{k+3} - \mu_k < 0$ for k sufficiently big and the derivative exists and vanishes. This is because by choosing a sufficiently small neighborhood $\{|x-a| < \delta\}$, we can guarantee that $k = k(x)$ is always greater than a given number for any x in this neighborhood. This concludes the proof of the theorem.

Note that, if $\mu_k \to \infty$ then the average of continued fraction elements of $\mathfrak{C}(a)$ tends to 1, and $\mathfrak{C}$ is not differentiable at $\mathfrak{C}(a)$. In other words, $\mathfrak{C}$ is almost surely not differentiable at $\mathfrak{C}(a)$. In the same vein, the derivative of $\mathfrak{C}$ at $a = [0, n, n, n, n, \dots]$ vanish for $n > 4$, and we see that $\mathfrak{C}$ is not differentiable at $\mathfrak{C}(a) = [0, 1_{n-1}, \overline{2, n-2}]$ or at best it will be of infinite slope at this point.

It might be of interest to study the derivatives at quadratic irrationalities in general. An exciting possibility would be to establish a connection with the derivative of $\mathfrak{C}$ and the Markov irrationalities, [2]. We don't know if the derivative may attain or not a finite non-zero value at some point.

7 Concluding remarks.

7.1 The Minkowski measure and the Denjoy measures.

The reader with a taste for singular functions such as $\mathfrak{C}$ will inevitably ask how it relates to Minkowski's question mark function. In our setting, this latter function is best understood as the cumulative distribution function of the unique $Aut_I(|\mathcal{F}|)$-invariant measure on the boundary $\partial\mathcal{F}$.

More generally[7], let $p \in (0, 1)$ be a real number. Suppose that a random walker on $\mathcal{F}$ starting at the edge I decides with probability $1/2$ to move through the positive sector of $\mathcal{F}$ and then each time he arrives to a bifurcation, he chooses to turn right with probability p and he chooses to turn left with probability $q := 1-p$. Similarly, he decides, with probability $1/2$ to move through the negative sector and then proceeds in the same manner at bifurcations. The probability that the walker ends up in the Farey interval $\mathcal{O}_{e'}$ is the product of probabilities of choices he makes to go from the base edge I to the edge e'.

Note that, unlike the classical random walkers, ours is a non-backtracking walker. He is not allowed to retrace his steps (so this is not a Markov process) [41]. Since by definition the topology of the boundary is generated by the Farey intervals, this puts a measure on the Borel algebra of $\partial\mathcal{F}_I$ and since the quotient map $\partial\mathcal{F}_I \to S^1_I$ is measurable with respect to Borel algebras, we obtain a probability measure μ_p on S^1_I, the *Denjoy measure* first introduced and studied by Denjoy [10]. Since $0 < p < 1$, the set of rationals is of measure 0, so that this passage to the quotient has no detectable effect on the measure space properties. Let $\mathbf{F}_p(x)$ be the cumulative distribution function of μ_p. Then $\mathbf{F}_{1/2}(x)$ is precisely the right extension of the Minkowski question mark function to $\widehat{\mathbf{R}}$ and provides

[7]In a yet more general setting, , take any function $P : V_\bullet(\mathcal{F}) \to [0, 1]$ and let the random walker, upon arriving to a vertex, choose to turn right with probability $P(v)$ and to turn left with probability $1 - P(v)$. Then this introduces a probability measure μ_P on $\partial\mathcal{F}$. A special class of measures is obtained, by requiring P to be only a function of the distance to the starting edge.

a canonical homeomorphism $S^1_I \to [0,1]/0\sim 1$ (recall that $\widehat{\mathbf{R}}$ is identified with S^1_I by the continued fraction map cfm and we may consider μ_p as a measure on $\widehat{\mathbf{R}}$ via this identification). As such, it conjugates the modular group to a group of homeomorphisms of $[0,1]/0\sim 1$. The functional equations for the question mark function (see Alkauskas' papers [4], [3]) then becomes a mere expression of this action. In fact, the conjugate representation of the modular group is a subgroup of Thompson's group T presented as the orientation preserving piecewise linear homeomorphisms of $[0,1]/0\sim 1$ with break points at dyadic rationals and such that the slopes of linear pieces are all powers of 2, see [20]. Now $\mathbf{F}_{1/2}(x)$ also conjugates $\mathfrak{z}$ to an involution of $[0,1]/0\sim 1$ (with jumps at dyadic rationals), and there are functional equations relating the action of this conjugate involution and the conjugate modular group action, similar to the ones we gave in the beginning of the paper. There is a possibility that this conjugate $\mathfrak{z}$ by the Minkowski function (or more generally by the functions $\mathbf{F}_p$) may interact in interesting ways with other singular functions such as the Takagi function[30]. See also the papers by Bonano and Isola [6], [5] for some more variations on this theme.

The measures μ_p are purely singular [52] with respect to the Lebesgue measure, i.e. their c.d.f's $\mathbf{F}_p$ have a vanishing derivative almost everywhere, a property shared by the involution $\mathfrak{z}$. In a different vein, it can be readily verified that the Minkowski measure[8] $\mu_{1/2}$ is an invariant measure for both the Gauss-Kuzming-Wirsing operator $\mathscr{L}_2^G$ (see the insightful paper of Vepstas [51]) and for the dual Gauss-Kuzming-Wirsing operator $\mathscr{L}_2^{\mathfrak{z}}$.

It seems that $\mathfrak{z}$ and $\mathbf{F}_{1/2}(x)$ recognize each other in another context: the integrals $\int \mathfrak{z}\, d\mathbf{F}_p(x)$ may well be finite.

7.2 Other trees and other continued fraction maps.

In fact, our story is about the group $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z} = \langle a, b \,|\, a^2 = b^3 = 1\rangle$ and a certain representation of this group as a subgroup of $\mathrm{PSL}_2(\mathbf{R})$, namely the representation whose image is the modular group. Every representation gives rise to a sort of continued fraction map, with particularly nice properties if the element ab is sent to a parabolic element. There are variations on this theme concerning the trees related to the free products $\mathbf{Z}/p\mathbf{Z} * \mathbf{Z}/q\mathbf{Z} \ldots$ and their representations in $\mathrm{PSL}_2(\mathbf{R})$. The stories are particularly nice if the targets of the representations lies inside $\mathrm{PGL}_2(\mathbf{Q})$. Hecke groups (see [28],[29]) are also promising in this respect. These stories involve various $\mathfrak{z}$-like automorphisms and boundary measures.

[8]The measure $\mu_{1/2}$ have also been called the *Lebesgue measure* of $\partial\mathcal{F}$ ([9]) and also the *Farey measure* [27].

7.3 Covariant functions

As we already mentioned in the introduction, the functional equations satisfied by $\zeta_{\mathbf{R}}$ says that it lax-covariant with the action of $\mathrm{PGL}_2(\mathbf{Z})$ on $\widehat{\mathbf{R}}$. Is it possible to find some analytic function on the upper half plane satisfying those functional equations (if necessary after modifying them by the complex conjugation)?

On the other hand, there are some analytic $\mathrm{PSL}_2(\mathbf{Z})$-covariant functions on the upper half plane, discovered by Brady [8] and Smart [46], see the papers by Basraoui and Sebbar for a recent account (they use the term "equivariant" instead of "covariant") [16] [43] [45]. The fact is that, if h is an equivariant function (i.e. if h satisfies the functional equations of ζ), then its Schwartzian is a weight-4 modular form. In the opposite direction, if f is a weight-k modular form, then the function

$$h(z) = z + k\frac{f(z)}{f'(z)}$$

is $\mathrm{PSL}_2(\mathbf{Z})$-covariant. The κ function of Kaneko and Yoshida [26] is covariant for an infinite subgroup of infinite index $\mathrm{PSL}_2(\mathbf{Z})$ (they use the word "covariant"). The Rogers-Ramanujan continued fraction $r(\tau)$ have some weak covariance properties, see [13]. Note that the identity function and the complex conjugation are $\mathrm{PSL}_2(\mathbf{Z})$-covariant, too.

We tried to concoct from $\zeta_{\mathbf{R}}$ a $\mathrm{PGL}_2(\mathbf{Z})$-invariant (singular) function on $\widehat{\mathbf{R}}$, but we failed.

7.4 Concepts similar to Jimm in Engineering

Digital communications implement some coding techniques to represent a data, and some signal conditioning transforms to obtain some desired properties on these representations. In analog communication when a carrier wave is used to transmit the data, as in the case of RF communications, a modulation of the carrier wave can be used to encode data; a sinusoidal carrier wave can be modulated in terms of its frequency, amplitude or phase. Be it RF, light, sound or electrical signals, the natural media used to transmit the digital information being analog, some discretizing mapping from digitally represented information to analog states is necessary.

Phase shift keying (PSK) is a modulation method to represent data in terms of changes of a sinusoidal carrier wave. When a modulation method assumes discrete values on modulated attribute of carrier wave, as in the case of digital communication, it is called "keying", hence the name phase shift keying. Binary PSK assumes two distinct states of 180 degrees separated phases with equal amplitude and frequency; simplest binary to BPSK mapping is to represent a 0 bit in digital as a specific phase and, a 1 bit as the inverted phase. All signal are subject to mostly arbitrary phase shift during transmission (except in very controlled media like length balanced wires or PCB traces), unless the source and the receiver has another information source to decide which of two phases is the neutral one (i.e. 0 bit), PSK modulation suffers from what's called phase ambiguity. Phase ambiguity can be addressed by supplying a non-modulated reference signal to compare the phase against (like a clock signal in digital electronics) which is not always practical; or by use of another transform on binary data called differential encoding. If a digital data is differentially encoded before PSK modulation, the each bit is not mapped to a specific phase of PSK states, but it is mapped to a specific change in phase. A differentially encoded data has the property called polarity insensitivity, which resolves the phase ambiguity issue (i.e. phone lines work no matter in which order the two copper wires are connected). Differential Manchester Encoding is a well known example of differential encoding which is used as the default encoding scheme of some versions of Ethernet (IEEE 802.3) computer network communications. Manchester encoding can also be thought of a digital counterpart of BPSK, it uses the digital clock signal to modulate the data signal via XOR operation; the resulting signal is same as the clock signal while the data bit (which is optionally differentially encoded before) is 0, and the inverse of clock signal if data bit is 1.

Jimm transformation of a number can be thought of being similar to BPSK; a number represented as zero and ones encoding the path on the farey tree, can be transformed using XOR operation to a modulation with a series of $(01)^\omega =$ 010101... which serves like a carrier signal or digital clock. We face the same phase ambiguity issue as in the case of BPSK, but in this case the problem can be solved by means of convention, such as by mapping the left turns to 0 and right turns to 1, and not the other way, for farey tree encoding, and the carrier signal to be $(01)^\omega$ and not to $(10)^\omega$ thus fixing the neural phase to be carrier series starting with 0. There is no invariant signal (number) under this transformation, as carrier signal contains 1s and XORing with 1 is equivalent to negation which is an unary operator with no fixed point; but the signal (number) closest to be minimally changing under this transformation is $(0011)^\omega$ giving

$$(0011)^\omega \veebar (01)^\omega = 011(0011)^\omega$$

which is the same signal 180 degrees out of phase, and the numbers related to $(0011)^\omega$ and $011(0011)^\omega$ are respectively $1+\sqrt{2}$ and $\sqrt{2}$.

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Appendix. Latex preamble commands for a latin-compatible typography of ج

```
\usepackage{arabtex}
\usepackage{adjustbox}
%Upper-case size jimm
\newcommand{\Jimm}{{\adjustbox{scale=.8, raise=0.7ex,%
trim=0px 0px 0px 7px, padding=0ex 0ex 0ex 0ex}{\RL{j}}}}
```

```
 %lower-case size jimm, to use as a substcript
\newcommand{\jimm}{{\adjustbox{scale=.5, raise=0.6ex,%
trim=0px 0px 0px 7px, padding=0ex 0ex 0ex 0ex}{\RL{j}}}}
```