Euler Re-summation of Multiplicative Series

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Abstract

...doesn't work. Which is a surprising result. The Euler transformation of alternating series is known to improve numeric convergence. Sometimes. Applied to the zeta-like series

$$\mathscr{M}_f(s) = \sum_{n=1}^{\infty} f_n^{-s}$$

for f_n a completely multiplicative arithmetic function, it fails. The intended question to be posed is: what classes of f_n result in $\mathscr{M}_f(s)$ obeying the Riemann hypothesis? A numerical survey addressing this question seems straightforward, if only the summation can be re-written to converge quickly in the critical strip. Euler re-summation is a basic, simple trick for achieving this. It works like a charm, for $f_n = n$, and utterly fails otherwise.

Introduction

A completely multiplicative sequence is an arithmetic function f_n taking values on the natural numbers n and being a homomorphism preserving the factorization of the integers: namely, $f_{nm} = f_n f_m$ holds. By convention, $f : \mathbb{N} \to \mathbb{C}$. Famous examples include $f_n = n^{-s}$ and $f_n = \chi(n)$ the Dirichlet characters. Of course, the divisor function and many other classical functions from number theory are known.

The Riemann hypothesis famously concerns the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and analogously the Dirichlet series

$$\mathscr{L}_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

How far can the hypothesis be extended? What other classes of sequences obey it? A natural conjecture is that it has something to do with the completely multiplicative nature of the series. Thus, an object worth consideration is the analogous series

$$\mathscr{M}_f(s) = \sum_{n=1}^{\infty} f_n^{-s} \tag{1}$$

given a completely multiplicative arithmetic function f_n. Several questions can be posed: Where are the zeros of $\mathscr{M}_f(s)$? What sort of functions f_n result in zeros on the critical line? How general is the setting for the Riemann hypothesis?

Recall that the homomorphism $f_{nm} = f_n f_m$ completely determines the value of f on composite integers; thus, a completely multiplicative function is completely specified by its values on the set of primes $\mathbb{P}$, i.e. by $f : \mathbb{P} \to \mathbb{C}$. It is not further constrained; there are uncountably many completely multiplicative functions.

Numerical exploration

Can we even get off the ground, here? Such a general setting is so broad that it's hard to find a place to start. Numerical exploration might provide quick, easy insights. Perhaps a simple place to start would be a perturbation of the primes

$$f_p = p\,(1+\varepsilon) \tag{2}$$

for some small (real or complex) ε.

Numerical exploration requires numerically stable convergent series. How might one find one? Given some generic sequence $f_p \geq 1$, it would seem likely that eqn. 1 (depending on the sequence) has a pole at $s = 1$. This obstructs naive summation; to get started, one needs some form of analytic continuation, or some re-summation that converges for $\Re s < 1$. The first obvious, simple trick is to create a conditionally convergent alternating series, analogous to the Dirichlet eta. It is easy to show the identity:

$$\mathscr{M}_f(s) = \frac{1}{1-2f_2^{-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{f_n^s}$$

where (depending on the sequence) the sum on the right might be expected to be conditionally convergent for $\Re s > 0$. As written, it is also clear that the rate of convergence is far too slow for numerical exploration.

The convergence of alternating series can often be improved by means of Euler summation. In this particular case, it seems promising to write

$$\mathscr{E}_f(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{f_n^s} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{f_{k+1}^s} \tag{3}$$

with the right-hand side being tame enough for numerical exploration.

Or so one might hope. The results surprised me.

- When $f_p = p$ then the sum converges quickly and easily. Using arbitrary-precision numerics, asking for varying degrees of precision, exploring the critical strip $0 < \Re s < 1$, there's no particular problem.
- When $f_p = p\,(1+\varepsilon)$ for $|\varepsilon| > 0$ the summations start out reasonably enough, and then becomes poor, verging on non-existent, soon offering no advantage at all over brute-force summation of the alternating series.

These statements can be made more precise. Consider the individual terms

$$t_n = \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{f_{k+1}^s}$$

For $\varepsilon = 0$, the re-summation is known to yield a globally convergent series for the Riemann zeta, as proven by Helmut Hasse in 1930; a modern treatment is given by Sondow[1], showing uniform convergence on compact sets. Numerically, what this means is that, for large enough n, that

$$t_n(\varepsilon = 0) \to \mathscr{O}\left(2^{-n}\right)$$

as a numerical observation (and not as an analytic claim; but the proof of uniform convergence says about as much.). Each term gets smaller by almost a factor of two. At that rate, it does not take particularly long to converge well. Convergence is exponential.

This does not happen for $|\varepsilon| > 0$. For the first few terms, one does see a similar behavior:

$$t_n(|\varepsilon| > 0) \sim t_n(\varepsilon = 0)$$

can be seen for a handful or few dozens of terms, depending on ε. This soon disappears, being replaced by

$$t_n(|\varepsilon| > 0) \sim \frac{1}{f_n^s} \tag{4}$$

That is, the Euler series transformation trick provides no acceleration at all.

That's very very interesting. This requires some thinking ....

Euler Transformation

A recap of the Euler transformation of series is in order. Given a convergent alternating series, the Euler re-summation is given by

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{k+1}$$

A simple visual derivation proceeds by re-summing with finite differences. One begins simply by rewriting:

$$a_1 - a_2 + a_3 - \cdots = \frac{a_1}{2} + \frac{1}{2}\left(\Delta a_1 - \Delta a_2 + \Delta a_3 - \cdots\right)$$

where $\Delta a_m = a_m - a_{m+1}$is the difference between successive terms. The expression in parenthesis is again an alternating series, so the re-summation is repeated. One defines the finite difference $\Delta^n a_m$ recursively as

$$\Delta^n a_m = \Delta^{n-1} a_m - \Delta^{n-1} a_{m+1}$$

terminating the recursion by $\Delta^0 a_m = a_m$. The re-summation is now

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=0}^{\infty} \frac{\Delta^n a_1}{2^{n+1}}$$

and it is straight-forward to invoke the binomial theorem to obtain the finite differences in terms of binomial coefficients

$$\Delta^n a_m = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{k+m}$$

A rich class of results can be obtained from algebraic re-arrangements, particularly when the a_n be be interpolated by a function, *viz.* $a_n = f(n)$ for some complex-analytic function $f(z)$ on the complex plane. In such a case, the re-summtion suggests a Newton series, which in turn link to the Newton-Mellin-Poisson cycle, as noted by Flajolet and Sedgewick.[2] This opens the path for the application of tools from analytic combinatorics.

The Newton interpolation formula is

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \binom{z-1}{n} \Delta^n a_1$$

but it is well-defined only when the finite differences are well-behaved. Returning to the perturbed prime sequence of eqn 2, the numeric evidence from eqn 4 indicates that

$$\Delta^n a_1 \sim \frac{2^n}{f_n^s}$$

which promptly overwhelms the binomial coefficient. The perturbed prime sequence does not have a Newton interpolation. This is easily seen for $z = 0$, as

$$(-1)^n \binom{-1}{n} = 1$$

or generally

$$(-1)^n \binom{-k}{n} = \binom{n+k-1}{n}$$

applies to $z-1 = -k$. For positive z, the situation is more subtle. In essence, the f_n are extremely "jagged", as a sequence – when n is a composite number, having many factors, it becomes combinatorially large: $f_n = n(1+\varepsilon)^{\Omega(n)}$ where $\Omega(n)$ is the number of prime factors of n, with multiplicity. When $\Omega(n)$ is large, so then f_n is far out of line from a placid linear progression, and the interpolant is forced to interpolate ever-wilder, spikier swings.

Conjectures

This suggests new questions: for which completely multiplicative functions f_n does a well-behaved Newton series exist? If the Newton series is well-behaved, then does it follow that eqn 3 is uniformly convergent on compact domains? And finally: if the re-summation is uniformly convergent on compact domains, does it then follow that the zeros lie in vertical strips, if not on vertical lines?

The use of the plural in the last conjecture follows from classical identities for the Liouville lambda $\lambda(n)$, which is completely multiplicative, and has a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{-s}} = \frac{\zeta(2s)}{\zeta(s)}$$

so that the corresponding critical line lies at $2s = 1 + i\tau$. Similarly, for the divisor function $\sigma_\alpha(n)$ generates the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^{-s}} = \zeta(s)\,\zeta(s-\alpha)$$

so that the zeros appear on two vertical lines. Note that the divisor function is multiplicative, but not completely multiplicative.

The first conjecture is too strong: the Newton series for the Liouville lambda and the divisor function are effectively undefined: the finite differences grow without bound. The second conjecture is much weaker than the first: for the Liouville lambda, the finite differences grow as

$$\Delta^n \lambda(1) \sim 2^n$$

and so it doesn't have a well-behaved Euler re-summation. But it does make it conceivable that an ill-behaved Newton series might still have a reasonable Euler re-summation: the inverse-power-of-two factor in the Euler re-summation can hide some amount of mis-behavior.

The conclusion seems to be that summability does not have a direct influence on the Riemann hypothesis: it seems to be a nice ingredient, but not a necessary one.

Conclusion

The Riemann hypothesis is a tough nut to crack, and part of that is that it is unclear where to search. The zeta function has a number of remarkable properties; but which of these, or which combination leads to a solution? One fairly evident conjecture is that "it has something to do with multiplicative sequences", which is what is being explored here. Easier said than done. There are an uncountable infinity of completely multiplicative functions, even if, as in the case of the Liouville lambda, one limits oneself to only two values.

References

[1] Jonathan Sondow, "Analytic continuation of Riemann's zeta function and values at negative integers via Euler's transformation of series", *Proceedings of the American Mathematical Society*, 120, 1994, pp. 421–424, URL http://www.ams.org/journals/proc/1994-120-02/S0002-9939-1994-1172954-7/S0002-9939-1994-1172954-7.pdf.

[2] Philippe Flajolet and Robert Sedgewick, "Mellin Transforms and Asymptotics: Finite Differences and Rice's Integrals", *Theoretical Computer Science*, 144(12), 1995, pp. 101–124, URL https://www.sciencedirect.com/science/article/pii/030439759400281M.