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Efficiency for Very Slow Assembly

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ABSTRACT

This report gives a qualitative description of some processes by which the multiplication in a slightly supercritical system may be stopped. It is shown that for a fixed mechanism of disassembly, whose effect is proportional to the energy release, the final efficiency, or total energy release, is roughly proportional to the α which the completed assembly would have in the absence of disassembling forces. Further, this energy is released in a small number of intense neutron bursts whose individual duration is short compared to the time required for the whole process. A rough quantitative estimate has been made on the assumption that disassembly is caused by thermal expansion of the core; a coefficient of expansion of the order 10^{-5} leads to reasonable values for the final efficiency, $\sim 10^{-9}$, and for the total duration, ~ 50 milliseconds. However, no other satisfactory mechanism of disassembly has been proposed to explain the case of a plutonium core, which is not believed to have a positive coefficient of expansion.

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Efficiency for Very Slow Assembly.

1. Differential equation for the efficiency

The following gives a qualitative description of some processes which may automatically stop multiplication in the slow assembly of a slightly supercritical system. The arguments used are essentially dimensional and no attempt at quantitative accuracy is made.

The "Assembly" of the supercritical system is characterized by two quantities: Firstly, the time variation of the multiplication constant $a_1(t)$. For a slow assembly (disregarding the effect of delayed neutrons) it may be assumed linear.

$$a_1(t) = at \tag{1}$$

where $t = o$ is the instant when the assembly is just critical. The second quantity is the number of neutrons at time $t = o$. If theassembly is not too slow - or the source which provides the initiating neutron is weak, the reaction is started essentially by a single neutron and the initial number of neuts is one. For very slow assembly or strong source, the density of neutrons builds up in the subcritical stage and depends on the constant a and the strength of the neutron source.

The "disassembly" will be characterized by the assumption that it contributes to the multiplication constant as follows

$$a_2(\phi) = \begin{cases} -b(\phi - \phi_o) & \text{for } \phi > \phi_o \\ o & \text{for } \phi < \phi_o \end{cases} \tag{2}$$

where ϕ is the fraction of fissionable atoms which have undergone fission. For example for thermal expansion affects we shall assume

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$$\phi_o = 0, \text{ i.e., } a_2 = -b\ \phi \ .$$

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A phase change would be characterized by a finite ϕ. The assumption that a_2 is a function of ϕ implies that the assembly is so slow that the "disassembling" forces have ample time to adjust the system to the given energy release. It differs in this respect radically from the fast assembly (as in the bomb) where the time scale for disassembly is determined by the multiplication constant a.

The total multiplication constant is defined by

$$a = (1/n)\ dn/dt = a_1(t) - a_2(\phi) \qquad (3)$$

where n is the density of neutrons.

If N is the number of fissionable atoms per cc, the number of fissions per unit time is given by $Nn\sigma_f v$, where σ_f is the fission cross section and v the neutron velocity. Thus

$$d\phi/dt = (1/N)\ dN/dt = \beta n; \quad \beta = \sigma_f v \qquad (4)$$

The effect of depletion of the number of fissions is negligible for our purposes. Hence $(1/N)\ (dN/dt)$ can be identified with $d\phi/dt$.

Combining (3) and (4) we find

$$\frac{d^2\phi / dt^2}{d\phi / dt} = a_1(t) - a_2(\phi) \qquad (5)$$

with the boundary conditions

$$\phi = 0, \quad d\phi / dt = \beta n_0 \ \text{ for } t = 0 \qquad (6)$$

Here n_0 is the density of neutrons at time $t = 0$.

It is clear that multiplication will stop only if the system is eventually subcritical. Thus, if for the completed assembly $a_1 = a_0$, we require that the final value of a_2 is larger than a_0.

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$$a_2'(\phi_f) > a_f$$

which gives immediately a lower limit for the final efficiency ϕ_f. For the linear law one has

$$\phi_f \geqslant \phi_o + a_f/b$$

Indeed we shall find that normally ϕ_f lies between the limits *

$$\phi_o + 2\, a_f/b \geqslant \phi_f \geqslant \phi_o + a_f/b$$

Thus for given b, the final efficiency is either proportional to a_f (with a numerical factor varying between 1 and 2) or it is equal to the "threshold efficiency" ϕ_o, depending on which is larger.

Consider first the very simplest case that the assembly is completed before any "disassembling" effect occurs. During the assembly we have

$$dn/dt = nat \qquad n = n_o\, e^{(1/2)\,at^2}$$

$$d\phi/dt = \beta n \qquad \phi = \beta n_o \int_0^t e^{(1/2)at^2}\, dt$$

After completion of the assembly we have (if there is no threshold)

$$d^2\phi/dt^2 = (d\phi/dt)\left[a_f - b\phi\right]$$

$$d\phi/dt = a_f\, \phi - (1/2)\, b\, \phi^2 + \text{const.}$$

The constant of integration depends on ϕ and $d\phi/dt$ at the instant when the assembly is complete. Since ϕ at this time increases rapidly, the constant will soon be negligible compared to ϕ.

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* There is one exception to this rule which is noted later.

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Hence $d\phi/dt = a_f\,\phi - (1/2)b\,\phi^2$

the end of the reaction is signified by the condition $d\phi/dt = 0$. This leads to the result

$$\phi_f = 2a_f/b$$

If there is a threshold efficiency, we find.

$$d\phi/dt = a_f\phi - (1/2)\,b(\phi - \phi_o)^2 + (1/2)\,b\,\phi_o^2$$

$$\phi_f = (1/b)\left[a_f + \sqrt{a_f^2 + b^2\,\phi_o^2}\right] \leq 2a_f/b + \phi_o.$$

This gives the upper limit for the final efficiency quoted above.

In the following these calculations will be refined. We shall consider the possibility that the disassembling factors become important before the assembly is complete. We shall see that this gives rise to a periodic process in which alternately the "assembly" predominates over the "disassembly" and vice versa. Normally this leads to a final efficiency dose to the lower limit quoted above.

2. Solutions for $\phi_o = 0$.

We consider the equation (5) first for the case that $\phi_o = 0$. Then, substituting (1) and (2)

$$\frac{d^2\phi/dt^2}{d\phi/dt} = at - b\phi \qquad (7)$$

Let us make the equation dimensionless by introducing

$$\tau = t\sqrt{a}$$

$$\chi = b\,\phi/\sqrt{a} \qquad (8)$$

then

$$\frac{d^2 x}{d\tau^2} = \frac{dx}{d\tau}\,(\tau - x)$$

$$x = 0, \quad dx/d\tau = (b/a)\,\beta n_0 = f, \quad \text{for } t = 0 \qquad (10)$$

The solutions of this equation are discussed in appendix A. Since the equation is invariant against a simultaneous translation in τ and x, the solutions can be expressed by a one-parameter family of functions. They are illustrated in Fig. 1.

The functions are all of the form

$$x = \tau + \text{periodic function.}$$

The simplest solution is

$$x = \tau \text{ for } f = (dx/d\tau)_0 = 1$$

If $f - 1 << 1$, we obtain a sinusoidal oscillation superposed over the function $x = \tau$ with period 2π. As f increases the sinusoidal oscillation is distorted and approaches eventually a step function with period $2\sqrt{2f}$.

Owing to the translational property of the differential equations these solutions may be displaced in the direction of the line $\varkappa = \tau$. If we displace the function shown in Fig. 1(c) so that the point A is placed at the origin, we get a function with a very small initial slope. The period in terms of the initial slope f is then approximately given by

$$2\sqrt{2 \log (4/f)}.$$

If $\varkappa = \tau$ the system is just critical. Solutions of type (a) and (b) would imply that the "disassembling" factors become important as soon as the system is critical.

This clearly will not occur in practical conditions. We should expect the system first to build up an appreciable neutron density in the supercritical state before "disassembly" can occur. i. e. $x - \tau$ should become negative. This occurs for the type (c) solution, if we start at the point A. From A to B and C the system is supercritical; from A to B the neutron density builds up rapidly; at B it becomes so large that its effect on the efficiency x becomes noticeable. At C it has become so strong that the "disassembly" overtakes the "assembly" and the system becomes subcritical; the neutron density falls again until at D it has become so small that no noticeable effect on x occurs; therefore the "disassembly" stops. From D to E to neutron density continues to fall until at E the "assembly" catches up again with the "disassembly" and the whole process is repeated.

The variation of neutron density n (α $dx / d\tau$) is shown Fig. 2 for the case (c). The duration of the neutron bursts is of the order $\tau' = 2\sqrt{2/\log(L/f)}$ which is a fraction $1/\log(L/f)$ of the period τ_1.

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3. Thermal expansion

The energy release in one fission is about 200 Mev. Assuming a specific heat of 3 k per atom, the temperature change is given by

$$T = \frac{200 \text{ Mev}}{3 k} \phi = 0.78 \times 10^{12} \phi \text{ degrees}$$

In actual fact it will be lower, since heat conduction into the tamper takes place.

Assuming a thermal expansion coefficient of the order 10^{-5}, the fractional change in liner dimensions is of the order $10^{7} \phi$.

The change in a should be equal to this quantity multiplied with the a of a highly supercritical system, which would be of the order of 10^{8}/sec. Hence we find that the constant b is about

$$b \backsim 10^{15}/\text{sec}.$$

a_f will be of the order of 1% of 10^{8}/sec; hence

$$\tau_f = a_f/b \backsim 10^{-9}$$

The temperature change to be expected is of the order 10^{3}.

We have to check whether there are really many oscillations within the time τ_f. For this purpose we require

$$\beta = \sigma_f v \backsim 2 \cdot 10^{-24} \text{ cm}^2 \cdot 1.4 \cdot 10^{9} \text{ cm/sec} \cong (1/3) \cdot 10^{-15} \text{ cm}^3/\text{sec}.$$

The time of assembly will be of the order of a mean free path (about 4 cm) divided by the velocity of the assembly. Assuming a drop from a height of a few cm under the force of gravity the velocity is of the order of 100 cm/sec

Thus $\tau_f \backsim 4 \cdot 10^{-2}$ sec

Then $a = a_f/\tau_f \backsim 2.5 \cdot 10^{7}/\text{sec}^2$

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The initial number of neuts will be one per core or about 1/400 pe

$$n_o \backsim 1/400\ cm^{-3}$$

Then

$$f = (b/a)\,\beta\, n_o = \backsim 3 \cdot 10^{-10}$$

The period in terms of the dimensionless variable τ is $2\sqrt{2\log(4/f)} \backsim 15$.

Hence period $= 2\sqrt{2\log(4/f)}\,/\sqrt{a} \backsim 3 \times \cdot 10^{-3}$

and is therefore about one tenth of the assembly time τ_f.

This explanation is satisfactory for a uranium-235 core but not for one of plutonium, which is believed to have a non-positive coefficient of thermal expansion. Some heat will be transmitted to the tamper but this cannot amount to more than one or two percent of the heat in the core (see Appendix B) and the consequent change of shape in the tamper will be of too small an order of magnitude.

From the theoretical point of view, we should expect that the estimate made above for the effect of such a gap on a is correct. However, a value derived from two near-critical experiments seems to indicate that the effect of a gap is about 10 times larger.

4. <u>Threshold Effect.</u>

We consider now the possibility that the "disassembly" starts only after a finite temperature has been reached. We may think for example of a phase change or of inhibition of thermal expansion in the tamper because the temperature is not uniform.

In either case the constant b would subsequently be larger. However, a much more important effect is, that the neutron density is allowed to increase unchecked until the threshold is reached. During this time we have

$$dn/dt = nat, \text{ hence } n = n_o\, e^{(1/2)\,at^2}$$

$$d\emptyset/dt = \beta n, \text{ hence } \emptyset = \beta n_o \int_o^t e^{(1/2)\,at^2}\,dt \cong (\beta n_o/at)\, e^{(1/2)\,at^2}$$

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The temperature according to section. 3 will be

$$T \backsim 10^{12} \; \phi = 10^{12} \, (\beta r_o / \sqrt{a}) \; \frac{e^{(1/2) a t^2}}{\sqrt{a} \; t} \backsim 1.5 \cdot 10^{-9} \; \frac{e^{(1/2) a t^2}}{\sqrt{a} \; t}$$

Assuming that the threshold is of the order of 100°, we require

$$\frac{e^{(1/2) \, a t^2}}{\sqrt{a} \;\; t} \backsim 10^{11} \quad \text{hence} \quad \sqrt{a} \;\; t \backsim 7.4$$

The neutron density has therefore been increased by a factor of the order 10^{10}. Introducing again x and τ we have initially (i.e. at the threshold)

$$\tau_o = \sqrt{a} \;\; t \backsim 7.4$$

Instead of x we should now use $x' = x - x_o$ (x_o is the threshold value). Then again $x_o' = 0$. However $dx/d\tau$ (which in the previous case was about 3×10^{-10}) has increased by a factor 10^{10} due to the increase in neutron density and an additional factor (let us call it R) because of the increase in b. Thus

$$(dx'/d\tau)_o \backsim 3R$$

is likely to be large.

As shown in the appendix, the value of f which determines the x-function to be used is in these conditions given by

$$f = (1/2) \;\; (x_o' - \tau_o)^2 + (dx'/d\tau)_o \backsim 27 + 3R$$

The period of the oscillation is

$$2 \; \sqrt{2f} \backsim \sqrt{54 + 6R}$$

(a change in the threshold temperature has very little effect on this result) whereas previously it was 15, and the assembly time was about 10 times as long. In order to bring the period up to the value of the assembly time we would have to assume an impossibly large value of R.

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($R \sim 4000$ corresponding for example to a change of 10% in dimensions). In these conditions the final efficiency is still determined by τ_f and we have the formula (13).

$$\phi_f' = \tau_f/b$$

However, b was assumed to be larger by the factor R. Thus the final efficiency would not be about $10^{-9}/R$. To this should be added the threshold efficiency ϕ_0, so that

$$\phi_f = \phi'_f + \phi_0 = (10^{-9}/R) + 10^{-10}$$

Thus the final efficiency could be down by a factor of the order 10, but the whole process remains essentially the same.

However, if $R > 10$, which seems reasonable, the efficiency is determined essentially by the threshold efficiency. It might be noted here that the first term ϕ_f' is very insensitive to the threshold temperature; the second term ϕ_0 of course is proportional to the threshold temperature.

5. The initial neutron density.

We assumed above without justification that initally there is just one neutron present. This implies that the system is not near critical for a sufficiently long time to produce an appreciable neutron density. Let us consider the system in the subcritical state, when

$$\alpha = at, \quad t < 0$$

If S is the neutron source per unit volume, we have

$$dn/dt = S + nat$$

with the solution

$$n = S\, e^{(1/2)at^2} \int_{-\infty}^{t} e^{-(1/2)\, at^2}\, dt$$

At $t = 0$ we have

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$$n_o = S \sqrt{\lambda/2a}$$

If S is due to contamination with plutonium-240, it will be about 300/sec cm^3. For a we found previous about 2.5 10^7 / sec^2. Then

$$n_o \sim 0.075/\text{cm}^3$$

which is larger than the value we assumed by about a factor 30. However, our results are very insensitive to the assumed value of n_o.

6. Effect of delayed neutrons.

The delayed neutrons make a difference of about 1% in criticality. If the velocity of assembly is v, a 1% change in criticality is obtained in a time of the order $\lambda/100v$, where λ is the mean free path.

The collision time for the delayed neutrons is of the order of 10^{-3} sec. The rate of change of a is proportional to the inverse collision time multiplied with the rate of change of criticality i.e.

$$a = da/dt = 10^3 \ v/\lambda$$

During the time t from the instant of criticality, the neutron density increases by the factor $e^{(1/2)\, a\, t^2}$. Hence during the time the system is supercritical for delayed neuts but subcritical for prompt neuts the increase in neutron density is about

$$e^{(1/20)\ \lambda/v}$$

These equations hold if the several parts which are far below critical, are assembled. However, if a piece is added to a nearly critical assembly, and if this piece makes only a small difference to the criticality of the system, say of the order of 1% (as assumed in the preceeding sections) then the rate of change of criticality decreases correspondingly and a should be decreased by about a factor 100, whereas t is increased by the same factor. Then the neutron

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density increases by the factor

$$e^{5\ 2/v}$$

For a velocity of assembly $V = 100$ cm/sec, this is negligible. For the velocity of assembly considered in the preceeding sections the delayed neutrons are therofore unimportant.

7. <u>The Limits for the Final Efficiency.</u>

If there is no threshold efficiency we have during the assembly the x-function shown in Fig. 2. This is repeated periodically until the assembly stops, say at $\tau = \tau_f$.

If this occurs during the level stretch of the disassembling stage, nothing further happens. The system remains subcritical and dies out. The efficiency ψ_f is then ~~somewhat~~ larger than τ_f but never more than by a factor 2. If the value $\tau = \tau_f$ gets us anywhere on the steep incline, say to the point B then the reaction in the steep incline is not affected, since in this region we assumed in any case that τ is constant. The reaction therefore proceeds into the level stretch and then stops,

If τ_f corresponds to the point C, we get the case treated previously, that the disassembly becomes important only after assembly is completed. It was shown there that in this case the final efficiency is $\psi_f = 2\tau_f$ and the reaction behaves as indicated by the dotted line. A similar situation exists at the point C'. In all cases it is obvious that the reaction remains always below the line $\psi = 2\tau$ and therefore $\psi_f = 2\tau_f$.

The situation is slightly more complicated if there is a finite threshold, because then the initial combinations (i.e. at the instant the threshold is reached) are not necessarily a zero of the ψ-function. Since initially the system is supercritical, we start off anywhere on the part of the ψ-curve where $x < \tau$. From this it follows that $x_f < 2\tau_f + \omega$, where ω is the period of the oscillation. Here x_f and τ_f are both measured from the threshold. If τ_o , x_o are the threshold values then $x_f < x_o + 2\tau_f + (\omega - 2\tau_o)$. This is the one exception to the rule that $\phi_o + 2a_f/b$. It is important only if $\omega - 2\tau_o$ larger than either of the first two terms.

Appendix A

We shall now consider the general behavior of the solutions of the differential equation

$$d^2x/d\tau^2 = (dx/d\tau) \quad (\tau - x) \tag{1}$$

Introducing

$$Y = x - \tau \tag{2}$$

it can also be written in the form

$$d^2Y/d\tau^2 + Y\,(1 + dy/d\tau) = 0 \tag{3}$$

If we multiply (3) with $(dy/d\tau) / \{1 + (dy/d\tau)$, we can integrate each term with the result $h\,(dy/d\tau) = dy/d\tau - \log(1 + dy/d\tau) = \text{const} - (1/2)\; y^2$

(4)

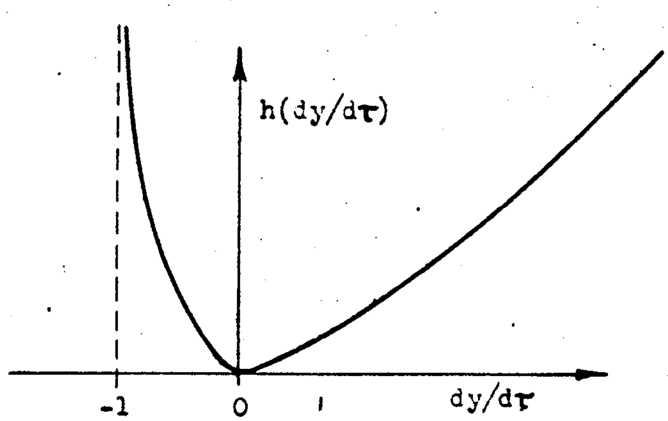

Fig. 3

The middle right-hand side, as function of $dy/d\tau$ is shown in Figure 3. It is infinite for $dy/d\tau = -1$, vanishes for $dy/d\tau = 0$ and then increases again to infinity for $dy/d\tau \rightarrow \infty$.

Hence it is always positive and therefore Y is bounded. We exclude in the following the trivial solutions $Y = 0$ or $X = \tau$; i.e. the constant in (4) is assumed finite and positive.

We shall prove next that Y is an oscillating function, with an infinite number of zeros. Assume that at some point Y and dy/dt are positive; then according to (4) h decreases (since y increases) and referring to the figure, it is found that dy/dt decreases and eventually reaches 0. It must do so in a finite time interval. The only alternative, that $dy/d\tau$ tends asymptotically to 0 leads to

$$y \rightarrow \text{finite value}$$
$$dy/d\tau \rightarrow 0$$
$$d^2y/d\tau^2 \rightarrow 0$$

which is in contradiction to (3). Indeed from (3) follows that d^2y/dt^2 is finite and negative for $y > 0$, $dy/d\tau = 0$ and therefore dy/dt becomes negative after a finite interval in τ. Then y decreases, h increases and therefore $dy/d\tau$ becomes more and more negative until y changes sign. Then the trend of $dy/d\tau$ is reversed and it follows as above that $dy/d\tau$ becomes positive again and after a while y becomes positive again, completing the circle.

The differential equation (3) remains unchanged if we shift the τ-scale. We can therefore demand without loss of generality that one of the zeros of y be at $\tau = 0$

$$y = 0 \quad \text{at} \quad \tau = 0 \tag{5}$$

From (3) and (5) follows immediately that y is antisymmetric

$$y(-\tau) = -y(\tau) \tag{6}$$

Let $\pm\tau_1$ be the first finite zeros of y. Now, because of the invariance of the differential equation against a translation, it follows that $y_1(\tau) = y(\tau - 2\tau_1)$ also satisfies the differential equation and of course $y_1(\tau_1) = y(-\tau_1) = y(\tau_1) = 0$.

Furthermore from (6)

$$(dy_1 / d\tau)_{\tau_1} = (dy/d\tau)_{-\tau_1} = (dy/d\tau)_{\tau_1}$$

Thus y_1 has at τ_1 the same value and the same derivative as y and is therefore identical with y. Thus

$$y_1(\tau) = y(\tau - 2\tau_1) = y(\tau) \qquad (7)$$

and therefore y is a periodic function.

In order to investigate the solutions, it will be convenient to go back to the function $x = \tau + y$. We demand again that $y = 0$ for $\tau = 0$ i.e.

$$x = 0 \quad \text{for} \quad \tau = 0 \qquad (8)$$

This can always be achieved by a simultaneous shift in the τ and x scale.

The solution is now completely determined, if we specify the derivative at $\tau = 0$; let it be f

$$dx/d\tau = f \quad \text{for} \quad x = 0. \qquad (9)$$

A trivial solution is obtained for $f = 1$; i.e.

$$x = \tau \quad \text{for } f = 1 \qquad (10)$$

This solution corresponds to $y = 0$. If f differs slightly from 1, y will be small and the second order term in (3) can be neglected. One finds

$$x = \tau + (f - 1) \sin \tau \quad \text{for} \quad |f - 1| << 1 \qquad (11)$$

Let us consider next the extreme case when f is large. Then for small τ we have $x = f\tau >> \tau$ and we may neglect τ in (1). We then have the equation

$$d^2x / d\tau^2 + x \, dx/d\tau = 0 \qquad (12)$$

Integration yields

$$dx/d\tau + (1/2) x^2 = \text{constant} = f \qquad (13)$$

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Hence

$$\tau = \int_0^x \frac{dx}{f - (1/2)x^2} = \sqrt{2/f} \quad \tanh^{-1} \quad (x/\sqrt{2f}) \qquad (15)$$

or

$$x = \sqrt{2f} \quad \tanh \quad (\tau \sqrt{f/2})$$

$$dx/d\tau = f \operatorname{sech}^2 (\tau \sqrt{f/2}) = f - (1/2)\, x^2 \qquad (15)$$

Let us define a value of $\tau = \tau'$ such that

$$\sqrt{2/f} \ll \tau' \ll 1$$

Since τ' is small the equations (15) apply. Since $\tau' \sqrt{f/2}$ is large we can use asymptotic formulae for the hyperbolic functions

$$x = \sqrt{2f} \quad \left[1 - 2\, e^{-\tau' \sqrt{2f}}\right]$$

$$dx/d\tau = 4f\, e^{-\tau' \sqrt{2f}} \qquad (16)$$

It appears therefore that x is practically constant and equal to $\sqrt{2f}$ in this range. We therefore solve the differential equation now for constant $x = x_1$, where x_1 must be near to $\sqrt{2f}$. In fact we shall see presently that $\sqrt{2f}$ is the correct value to be chosen.

Then

$$d^2x/d\tau^2 = (dx/d\tau)\ (\tau - x_1)$$

$$dx/d\tau = \text{constant} \cdot e^{(1/2)(\tau - x_1)^2}$$

The constant can be determined by fitting to the expression (16) at $\tau = \tau'$. Then

$$dx/d\tau = 4f\, e^{-\tau'\sqrt{2f} - (1/2)(\tau' - x_1)^2 + (1/2)(\tau - x_1)^2}$$

$$= 4f\, e^{(1/2)(\tau - x_1)^2 - (1/2) x_1^2 - (1/2)\tau'^2 + \tau' (x_1 - \sqrt{2f})}$$

This should be independent of the value τ' chosen. ... Since τ' is small $x_1 = \sqrt{2f}$, we have in first approximation

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$$dx/d\tau = 4 f e^{(1/2)(\tau - x_1)^2 - (1/2) x_1^2} \qquad (17)$$

integrating again and fitting at $\tau = \tau'$ we find

$$x = \sqrt{2f}\,\left[1 - 2e^{-\tau'\sqrt{2f}}\right] + \int_{\tau'}^{\tau} 4 f e^{(1/2)(\tau - x_1)^2 - (1/2) x_1^2}\, d\tau \qquad (18)$$

We write this in the form

$$\psi = -\sqrt{2f} + 4f e^{-(1/2) x_1^2} \int_0^{\tau} e^{(1/2)(\tau - x_1)^2}\, d\tau \qquad (19)$$

plus an additional term, which we shall show to be negligible. The additional term is

$$2\sqrt{2f}\left[1 - e^{-\tau'\sqrt{2f}}\right] + 4f e^{-(1/2) x_1^2} \int_0^{\tau'} e^{(1/2)(\tau - x_1)^2}\, d\tau$$

Since τ in the integrand is small, we may neglect τ^2 in the exponent and find

$$2\sqrt{2f}\left[1 - e^{-\tau'\sqrt{2f}}\right] + 4f \int_0^{\tau'} e^{-\tau x_1}\, d\tau = 2\sqrt{2f}\left[1 - e^{-\tau\sqrt{2f}} + (\sqrt{2f}/x_1)(e^{-\tau' x_1} - 1)\right]$$

which vanishes, since $x_1 = \sqrt{2f}$.

We may write the formula (19) in the following form

$$x = -\sqrt{2f} + 4f e^{-(1/2) x_1^2} \int_0^{x_1} e^{(1/2)(\tau - x_1)^2}\, dx_1 - 4f e^{-(1/2) x_1^2} \int_{\tau}^{x_1} e^{(1/2)(\tau - x_1)^2}\, d\tau \qquad (20)$$

Since x_1 is large we may use the asymptotic expression for the integral and find

$$x = \sqrt{2f}\left[2\sqrt{2f}/x_1 - 1\right] - 4f e^{-(1/2) x_1^2} \int_{\tau}^{x_1} e^{(1/2)(\tau - x_1)^2}\, d\tau \qquad (21)$$

It is easily seen that the second term is small compared to the first, provided

$$\tau \gg 1/\sqrt{2f} \quad \text{and} \quad 2x_1 - \tau \gg 1/\sqrt{2f}$$

The assumption on which our approximation is based is therefore justified until τ comes close to $2x_1$. (This argument would not pass a mathematical

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could be refined). Apart from the first constant factor, x is antisymmetric in $(\tau - x_1)$. From the discussion of the function y given above it follows that x_1 must be a zero of the function y, i.e. $x = x_1$ for $\tau = x_1$. This leads to the condition

$$x_1 = \sqrt{2f} \left[2\sqrt{2f}/x_1 - 1 \right] \qquad (22)$$

which has the solution

$$x_1 = \sqrt{2f} \qquad (23)$$

If we carry the asymptotic expression for the integral one step further, it is found that x_1 differs from $\sqrt{2f}$ by a term of the order $1/\sqrt{f}$. This justifies our treatment, since we made repeatedly use of the face that $x_1 - \sqrt{2f} << 1$.

Because of the antisymmetry and periodicity of $y = \tau - x$ we now have the complete solution of the problem. As τ approaches $2x_1$ the function x behaves in the same way as near $\tau = 0$ except that τ has been shifted by the amount $2x_1$. The general behavior of x is indicated in Fig. 4. In the first rapidly rising parts x is given by (19), i.e. $x = \sqrt{2f} \tanh(\tau\sqrt{f/2})$

$$dx/d\tau = f - (1/2)\, x^2 \qquad (24)$$

$|\tau| << 1$

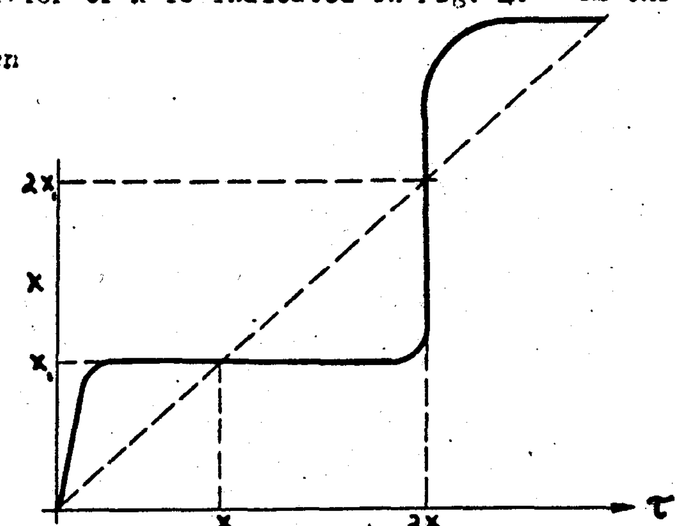

Fig. 4.

In the second steep rise the same equations hold except that τ and x are both shifted by the amount $2x_1$.

In the level part we have the equations (17) and (20), which can be written in the form

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[redacted]

[redacted]

$$x = x_1 + 4f\, e^{-(1/2)\, x_1^2} \int_0^{\tau} e^{(1/2)\xi^2}\, d\xi$$

$$dx/d\tau = 4f\; e^{-(1/2)\, x_1^2}\; e^{(1/2)(\tau - x_1)^2}$$

$$\sqrt{2/f} \ll \tau; \quad 2x_1 - \tau \gg \sqrt{2/f} \tag{25}$$

$$x_1 = \sqrt{2f}$$

Let us now shift the solution by only half a period. Then the point $\tau = x = x_1$ is shifted to $\tau = x = 0$ and we obtain another solution of our problem in which the initial slope at $\tau = 0$ is very small. From (25) follows that at $\tau = 4$, we have $dx/d\tau = 4f \exp(-f)$. If we define

$$f' = 4f \exp(-f) \tag{26}$$

It follows that the solution $x_{f'}$ is identical with the solution x_f except for the indicated shift of coordinates.

Thus we have also obtained the solution if the initial slope is very small. Negative slopes are excluded, since $dx/d\tau = 1 + dy/d\tau > 0$ according to (4). (More precisely $dx/d\tau$ is either always positive, or always negative; but the second case is of no physical interest).

There remains the problem of determining the value of f to be chosen for given initial conditions. In general we shall be interested in problems in which x and $dx/d\tau$ are given for some value of τ. Allowing for a simultaneous shift in x and τ, the significant quantities are

$$y = x - \tau$$

$$x = dx/d\tau \tag{27}$$

We have to find f for given x and y. We consider only the case when f is large.

If y = 0 and x >> 1, we have the case treated above and clearly we choose x = f, $x = \tau = 0$. We then are on the fast rising part of the curve, where the equations (24) apply. As long as we are on this part of the

[redacted]

curve τ will be negligible and we can in the general case identify x with y. Then from (24)

$$x = y = \sqrt{2f}\ \tanh\ (\tau \sqrt{f/2})$$

Thus

$$dx/d\tau = x = f - (1/2)\ y^2$$

$$f = x + (1/2)\ y^2$$

$$\tau = \sqrt{2/f}\ \tanh^{-1}\ (y/\sqrt{2f}\ )$$

$$x = \tau + x \qquad (28)$$

The equations are applicable provided $f >> 1$ and $\tau << 1$. This leads to the conditions

$$x + (1/2)\ y^2 >> 1$$

$$x >> 2y^2\ e^{-|y|}$$

These conditions map out the region I in Fig. 5.

If we are on the level part of the x-curve, we would normally expect $x = dx/d\tau$ to be small. If we are just at one of the zeros of the y-function, we have $y = 0$, $x << 1$, and therefore the region II in which this part of the x-curve applies extends also to the x-axis, but for small values of y, as shown in Fig. 5.

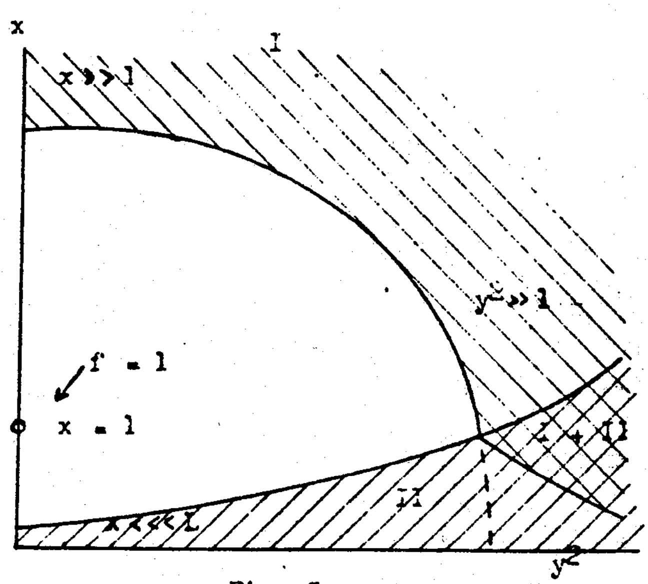

Fig. 5

Applying the equations (25), we find

$$x = x_1 = \sqrt{2f}$$

$$x - \tau = y = \sqrt{2f} - \tau$$

[redacted]

$$dx/d\tau = x = 4f\, e^{-f}\, e^{(1/2)(\tau - x_1)^2} = 4f\, e^{[illegible]}\, e^{(1/2)\, y^2}$$

If y is negative, it is convenient to use negative values of τ and $x = x_1$. The equations remain the same except for suitable changes of sign. The above equations give

$$e^f/f = 4\, e^{(1/2)\, y^2/x}$$

$$\tau = \sqrt{2f} - y \qquad (29)$$

$$x = \sqrt{2f}$$

These equations hold provided f is large, which requires

either $y >> 1$ or $x <<< 1$

($x <<< 1$) is intended to indicate that $\log (1/x)$ should still be large, but not necessarily $\log \log 1/x$.

Furthermore we have to exclude very small values of τ. Since f is large and $\tau_2 \sqrt{2f} - y$, this limitation will occur only for large values of y. One can see easily that $\tau = 0$ for $f = x = (1/2)\, y^2$. Hence the region II looks roughly as indicated in Fig. 5.

The region left blank in Fig. 5 corresponds to values of f of the order 1. $f = 1$ corresponds to $y = 0$, $x = 1$.

Appendix B.

Thermal Expansion of Tamper

The heat lost by the core to the tamper will certainly be less (in fact very much less) than that lost to a surrounding perfect conductor. In a short time t this functional loss is $(3a/R)\sqrt{t/\pi}$ where R is the radius of the core and a the thermal diffusivity.

If $t = \tau_f \sim 3.10^{-2}$ sec, and we assume $a \sim 10^{-1}$ this is of the order 10^{-2}; thus the heating of the tamper by conduction is negligible compared to the heat imported by radiation which will be of the order of one percent of the heat developed in the core.

The heating of the tamper will produce thermal stress tending to expand it; we can calculate the stresses from formulae given by Timoshenko in his "Theory of Elasticity". If we consider the tamper 'thin' the others produced in a hemisphere will be equivalent to a couple applied to its rim, and generalizing from well-known formulae relating the applied couple to the change in curvature, we find that the relative change in curvature of the tamper will be

$$\frac{\Delta R}{R} \sim \frac{3}{2} \frac{\alpha \int \Delta T \, dx}{d}$$

where $\int \Delta T \, dx$ is the integral through the shell of the change in temperature and d is its thickness.

Now the heat gained by the shell is

$$4\pi R^2 \rho_T c_T \int \Delta T \, dx$$

and this must be equal to that lost by the core,

$$q(4/3)\pi R^3 \rho_c c_c T_0 \qquad [q \sim 10^{-2}]$$

so that

$$\int \Delta T \, dx \sim q T_0 (\rho_c c_c/\rho_T c_T) R/3$$

and

$$(\Delta R)/R \sim (R/2d)(\rho_c c_c/\rho_T c_T) q (\alpha T_0)$$

Now

$$\rho_c c_c/\rho_T c_T \sim 1/3 \qquad \text{so that } \Delta R/R \sim q(\alpha T_0)$$

assuming $R \sim 6d$.

Thus the thermal expansion of the tamper is a fraction q of that produced in the core if the latter had the same coefficient of expansion. $q \sim 10^{-2}$ this would lead to the conclusion that an expansion of the tamper is not sufficient to stop the assembly in the manner suggested.

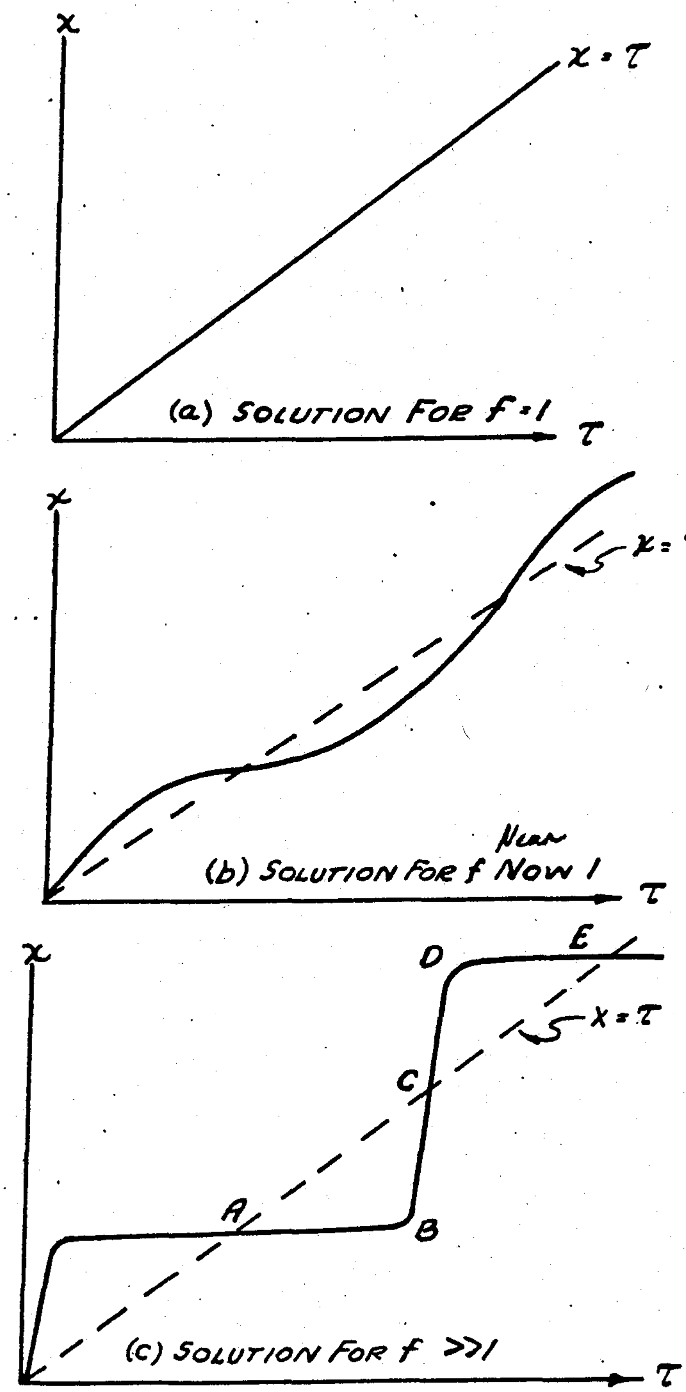
x
x = τ
(a) Solution For f = 1
τ
x
x = τ
Near
(b) Solution For f Now 1
τ
x
D
E
x = τ
C
A
B
(c) Solution For f >>1
τ

FIG. 1

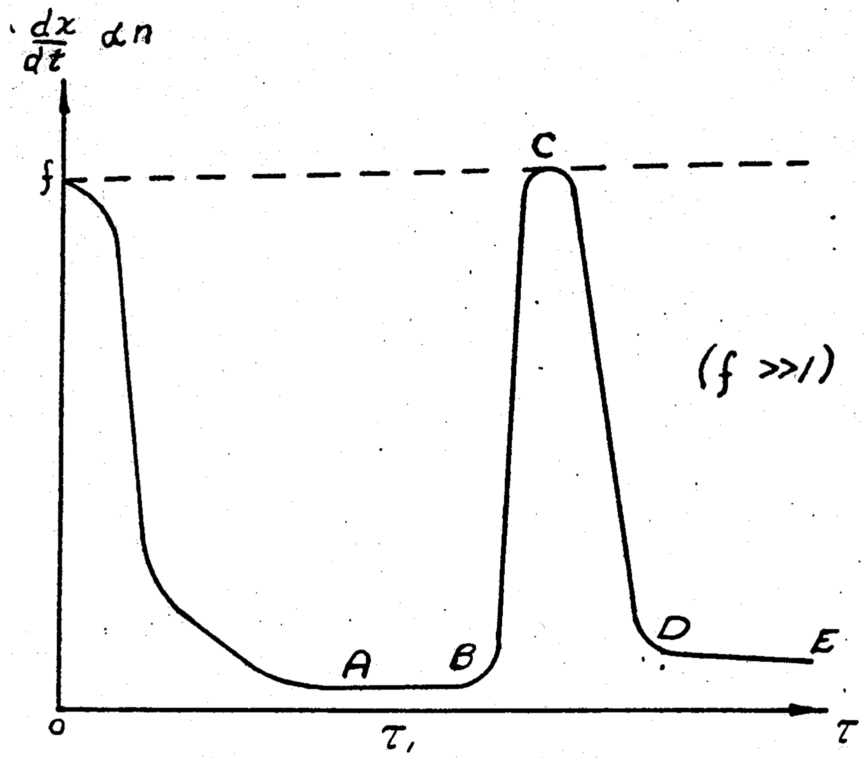

FIG. 1d

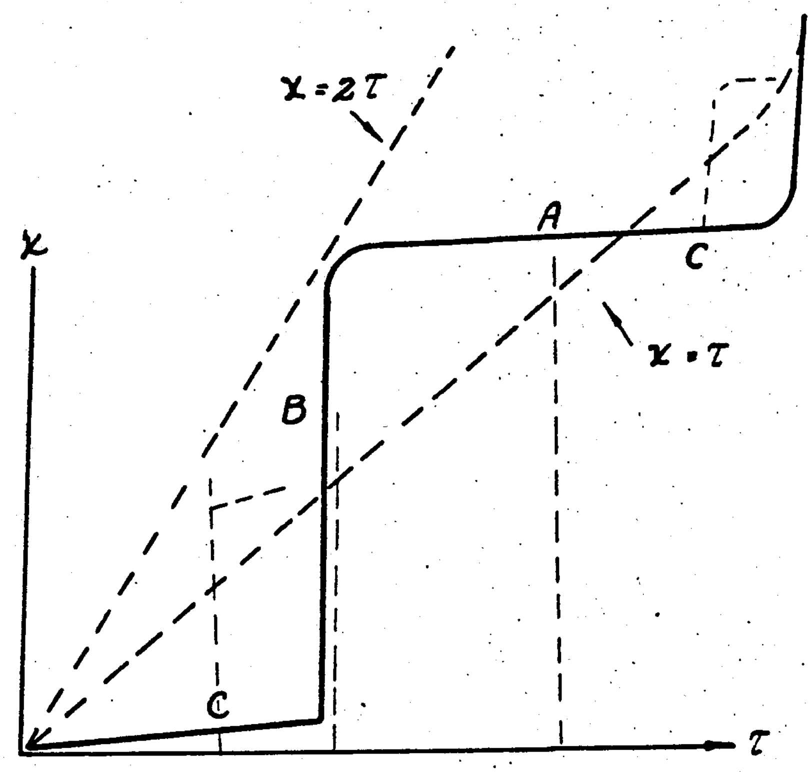

FIG. 2